

Free upper boundary value problems for the semi-geostrophic equations

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June 27, 2016

Abstract

The semi-geostrophic system is widely used in the modelling of large-scale atmospheric flows. In this paper, we prove existence of solutions of the incompressible semi-geostrophic equations in a fully three-dimensional domain with a free upper boundary condition. The main structure of the proof follows the pioneering work of Benamou and Brenier [7], who analysed the same system but with a rigid boundary condition. However, there are very significant new elements required in our proof of the existence of solutions for the incompressible free boundary problem. The proof uses on optimal transport results as well as the analysis of Hamiltonian ODEs in spaces of probability measures given by Ambrosio and Gangbo [5]. We also show how these techniques can be modified to yield the analogous result for the compressible version of the system.

1 Introduction

The fully compressible semi-geostrophic system, posed in a domain of the form $[0, \tau) \times \Omega$, with $\Omega \subset \mathbb{R}^3$ a bounded subset of the physical space, is the following system of equations:

$$D_t \mathbf{u}^g + f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u} + \nabla \phi + \frac{1}{\rho} \nabla p = 0, \quad (1.1)$$

$$D_t \theta = 0, \quad (1.2)$$

$$D_t \frac{1}{\rho} = \frac{1}{\rho} \nabla \cdot \mathbf{u}, \quad (1.3)$$

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \frac{1}{\rho} \nabla p = 0, \quad (1.4)$$

$$p = R\rho\theta \left(\frac{p}{p_{\text{ref}}} \right)^{\frac{\kappa-1}{\kappa}}, \quad (1.5)$$

where D_t denotes the lagrangian derivative operator:

$$D_t = \partial_t + \mathbf{u} \cdot \nabla \quad (1.6)$$

The unknowns in the above equations are $\mathbf{u}^g = (u_1^g, u_2^g, 0)$, $\mathbf{u} = (u_1, u_2, u_3)$, p , ρ , θ ; we assume R , f_{cor} and p_{ref} constant, and indeed we will assume $f_{\text{cor}} = 1$ in what follows. We also assume $\Phi(\mathbf{x}) = g_{\text{grav}} x_3$. The physical significance of each variable is given in the Appendix.

This system is obtained as an approximation to the laws of thermodynamics and to the compressible Navier-Stokes equations, the fundamental equations that describe the behaviour of the atmosphere, or more precisely the version obtained when viscosity is neglected, known as the Euler equations. The particular approximation made in the derivation of the semi-geostrophic system is valid on scales where the effects of rotation dominate the flow. In this case, the effect of the Coriolis and of the pressure gradient force are balanced, and equation (1.4) is precisely a formulation of hydrostatic and geostrophic balance. The remaining equations formulate other physical properties: (1.1) is the momentum equation; (1.2) represents the adiabatic assumption; (1.3) is the continuity equation and (1.5) is the equation of state which relates the thermodynamic quantities to each other.

The semi-geostrophic system was first introduced by Eliassen [17] and then rediscovered by Hoskins [21]. It admits more singular behaviour in the solutions than other reductions with a simpler mathematical structure, such as the quasi-geostrophic system, and for this reason this system has been used in particular to describe the formation of atmospheric fronts.

For an accurate representation of the behaviour of large-scale atmospheric flow, one should consider the fully compressible semi-geostrophic equations with variable Coriolis parameter and a free upper boundary condition. The complexity of this problem means that so far results have only been obtained after relaxing one or more of these conditions. We give a brief summary of these results.

In [7], Benamou and Brenier assumed the fluid to be incompressible, the Coriolis parameter constant and the boundaries rigid. The problem they considered, written in dimensionless scalar form, is posed in a fixed domain $\Omega \subset \mathbb{R}^3$ and given by

$$\begin{cases} D_t u_1^g - u_2 + \frac{\partial p}{\partial x_1} = 0, \\ D_t u_2^g + u_1 + \frac{\partial p}{\partial x_2} = 0, \\ D_t \rho = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \frac{\partial p}{\partial x_1} = u_2^g, \quad \frac{\partial p}{\partial x_2} = -u_1^g, \quad \frac{\partial p}{\partial x_3} = -\rho. \end{cases} \quad (t, x) \in [0, \tau) \times \Omega \quad (1.7)$$

The equations are to be solved subject to appropriate initial conditions, and the rigid boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad (t, x) \in [0, \tau) \times \partial\Omega, \quad (1.8)$$

where $\partial\Omega$ represents the boundary of Ω and \mathbf{n} is the outward unit normal to $\partial\Omega$.

Using a change of variables, first introduced by Hoskins in [21], one derives the so-called *dual formulation* of the system, that elucidates the Hamiltonian structure of the problem. Indeed, in this formulation, the equations are interpreted as a Monge-Ampère equation coupled with a transport problem, and this elegant interpretation yields the proof of the existence of weak solutions of the system in dual space, based on the groundbreaking work of Brenier [8].

This result was generalised in [15] to prove existence of weak solutions for the 3-dimensional compressible system (1.1)-(1.5), still assuming a fixed boundary and a rigid boundary condition.

In [13], Cullen and Gangbo relaxed the assumption of rigid boundaries assuming a more physically appropriate free boundary condition. However, they made the additional assumption of a constant potential temperature, and thus obtained a 2-D system, known as the semi-geostrophic shallow water system, posed on a fixed two-dimensional domain. After passing to dual variables, they showed existence of weak solutions of the resulting dual problem.

The above results were obtained for the dual space formulation of the equations, which is the setting we also consider in the present paper. However, we mention for completeness more recent results regarding the existence of solutions in the original physical variables. The first step in this direction was taken by Cullen and Feldman, who proved in [12] the existence of Lagrangian solutions in physical variables, a result that was extended in [14] to the compressible system. Recently, Ambrosio et al have succeeded in proving existence of solutions for the Eulerian formulation, in cases when there are no boundary effects [3, 4].

In this paper, we extend the results above to prove the existence of dual-space solutions for the *incompressible system, in three-dimensional space, in a domain with a free upper boundary*. This result is stated in Theorem 3.6, and is a direct but substantial extension of the results of Cullen and Gangbo. The proof differs from the one given in [13] also in its use of the approach introduced in [14], namely it exploits the general theory of Hamiltonian ODEs in spaces of probability measures given in [5]. The strategy of the proof is to show that the Hamiltonian of the system, given by the dual energy, satisfies the necessary conditions to invoke the general theory of [5], and that its superdifferential coincides precisely with the dual velocity of the flow. This, coupled with the existence of the optimal transport map for the given cost function, yields the desired result. We also sketch the extension of this proof to the compressible case. Namely, by writing the equations in pressure coordinates, we extend the result of [15], who considered the compressible equations but assumed rigid boundary conditions, to the more physically relevant case of free boundary conditions.

We mention that recently Caffarelli and McCann [9] have developed extensively a general theory of optimal transport in domains with free boundaries. It would be interesting to verify whether these general results can be used to give an alternative proof of the problem considered here.

The paper is organised as follows:

In Section 2, we summarise the results of Benamou and Brenier on the solution of the incompressible 3-D system in dual space, with rigid boundary conditions. The proof of this result sets the strategy for all generalisations, and we highlight how our approach differs from this.

In Section 3, we consider the same problem but assume a more realistic free boundary condition on the top boundary (the surface of the fluid). We first summarise the results for the 2-D case obtained by Cullen and Gangbo, then give the proof for the 3-D case. This is the main result of this paper.

In Section 4, we extend the results to the compressible system. In view of the fact that, in pressure coordinates, the two problems are formally identical, this extension does not introduce any new element.

In the Appendix, we list various definitions and the notation we use throughout, as well as some general results in the theory of optimal transport and Hamiltonian flows that we appeal to in the proof of our results.

2 The incompressible semi-geostrophic system in a fixed domain

We start by describing the strategy common to proving the existence of solutions, in a particular set of coordinates, in all cases we examine. The original approach is due to Benamou and Brenier [7].

Let $\Omega \subset \mathbb{R}^3$ be a fixed bounded domain, and $\tau > 0$ a fixed constant. Consider the system of equations (1.7), with suitable prescribed initial conditions and the rigid boundary conditions

given by (1.8).

The geostrophic energy, which is conserved by the flow, is given by

$$E = \int_{\Omega} \left(\frac{1}{2} ((u_1^g)^2 + (u_2^g)^2) + \rho x_3 \right) d\mathbf{x}. \quad (2.1)$$

An important physical property of the flow described by the semigeostrophic approximation is summarised in the following fundamental principle.

Principle 2.1 (Cullen’s stability principle). *Stable solutions of (1.7)-(1.8) correspond to solutions that, at each fixed time t , minimise the energy E given by (2.1) with respect to the rearrangements of particles, in physical space, that conserve the absolute momentum $(u_1^g - x_2, u_2^g + x_1)$ and the density ρ .*

This was expressed in [24] as the requirement that states corresponding to critical points of (2.1) with respect to such rearrangements of particles in physical space are states in hydrostatic and geostrophic balance. The evolution of states that are critical points of the energy but not minima cannot be described by the semi-geostrophic approximation [11]. The significance of Brenier’s work is in the elucidation of the precise mathematical meaning of this minimisation principle, and its mathematical formulation in the framework of convex analysis and optimal transport theory. This machinery can be used after a change of variables, introduced by Hoskins [21] and motivated by physical considerations. In these variables, the problem is formulated mathematically in Hamiltonian form, and the time evolution of the velocity is expressed explicitly.

Formulation in dual variables

The change to dual coordinates $\mathbf{y} = \mathbf{T}(t, \mathbf{x})$ is defined by

$$\mathbf{T} : \Omega \rightarrow \mathbb{R}^3 : \quad T_1(\mathbf{x}) = x_1 + u_2^g, \quad T_2(\mathbf{x}) = x_2 - u_1^g, \quad T_3(\mathbf{x}) = -\rho. \quad (2.2)$$

Note that (1.7) implies

$$(y_1 - x_1, y_2 - x_2, y_3) = \nabla p.$$

The energy functional (2.1) is formulated in dual variables as

$$E(t, \mathbf{x}, \mathbf{T}) = \int_{\Omega} \left(\frac{1}{2} \{ |x_1 - T_1(\mathbf{x})|^2 + |x_2 - T_2(\mathbf{x})|^2 \} - x_3 T_3(\mathbf{x}) \right) d\mathbf{x}. \quad (2.3)$$

The geostrophic coordinates are related to Cullen’s stability principle through the so-called *geopotential* $P(t, \mathbf{x})$, defined as

$$P(t, \mathbf{x}) = \frac{1}{2} (x_1^2 + x_2^2) + p(t, \mathbf{x}). \quad (2.4)$$

One can perform a formal variational computation, with respect to variations φ of particle position satisfying the incompressibility constraint $\nabla \cdot \varphi = 0$ and that conserve absolute momentum so that $u_1^g - \varphi_2 = u_2^g + \varphi_1 = 0$. This computation indicates that, for the energy in (2.3) to be stationary, it must hold that $\mathbf{T}(\mathbf{x}) = \nabla P$, and that the condition for the energy to be minimised is that $D^2 P$ is positive definite, where D^2 is the Hessian. Positive definiteness of $D^2 P$ implies that P is convex, see [11, 16, 20, 24]. Hence the stability principle can be formulated as a convexity principle.

Principle 2.2 (Cullen's convexity principle). *Minima of the energy (2.1), with respect to variations as in Principle 2.1, correspond to a geopotential $P(t, \mathbf{x})$, as given by (2.4), which is a convex function of \mathbf{x} .*

We can now express the dual formulation in the language of optimal transport theory, [6, 25].

Definition 2.1. *The potential density $\nu(t, \mathbf{x}) \in \mathcal{P}([0, \tau] \times \Omega)$ associated to the system (1.7) is the push forward of the Lebesgue measure of the domain Ω through the map \mathbf{T} given by (2.2):*

$$\nu = \mathbf{T} \# \chi_\Omega. \quad (2.5)$$

This means that the measure ν is defined by

$$\nu(B) = |\mathbf{T}^{-1}(B)|, \quad \forall B \subset \mathbb{R}^3 \text{ Borel set},$$

and satisfies the change of variable formula

$$\int_\Omega f(\mathbf{T}(\mathbf{x})) d\mathbf{x} = \int_{\mathbb{R}^3} f(\mathbf{y}) d\nu(\mathbf{y}) \quad \forall f \in \mathbf{C}_c(\mathbb{R}^3).$$

We can now rephrase Cullen's stability principle as the requirement that \mathbf{T} which minimises (2.3) is the *optimal map* in the transport of χ_Ω to ν with respect to the cost function $c(\mathbf{x}, \mathbf{y})$ given by

$$c(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \{|x_1 - y_1|^2 + |x_2 - y_2|^2\} - x_3 y_3. \quad (2.6)$$

Brenier's polar factorization theorem [8] ensures the existence of a unique such optimal map, and guarantees that this optimal map, for each fixed time t , is of the form $\mathbf{T} = \nabla P$ with P a convex function of the space variable \mathbf{x} .

Hence defining \mathbf{T} as in (2.2) and P as in (2.4), we can use the fact that $D_t \mathbf{x} = \mathbf{u}$, to rewrite (1.7)-(1.8) as the following system of equations for $P(t, \mathbf{x})$, $\mathbf{u}(t, \mathbf{x})$:

$$D_t \mathbf{T}(t, \mathbf{x}) = J(\mathbf{T}(t, \mathbf{x}) - \mathbf{x}), \quad (2.7)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.8)$$

$$\mathbf{T}(t, \mathbf{x}) = \nabla P(t, \mathbf{x}), \quad (2.9)$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } [0, \tau] \times \partial\Omega, \quad (2.10)$$

with initial condition

$$P(0, \mathbf{x}) = P_0(\mathbf{x}) := \frac{1}{2}(x_1^2 + x_2^2) + p_0(\mathbf{x}) \text{ in } \Omega, \quad (2.11)$$

where the symplectic matrix J is defined by

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.12)$$

We now write (2.7)-(2.11) in Lagrangian form. We define the Lagrangian flow map $\mathbf{F}(t, \mathbf{x})$ corresponding to the velocity \mathbf{u} , i.e.

$$\frac{\partial}{\partial t} \mathbf{F}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{F}(t, \mathbf{x})), \quad \mathbf{F}(0, \mathbf{x}) = \mathbf{x},$$

and can then rewrite (2.7), (2.9), as first done in [12], in the form

$$\frac{\partial}{\partial t} \mathbf{Z}(t, \mathbf{x}) = J(\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})), \quad \mathbf{Z}(t, \mathbf{x}) = \nabla P(t, \mathbf{F}(t, \mathbf{x})). \quad (2.13)$$

The incompressibility condition and the boundary condition can then be reformulated as

$$\mathbf{F}(t, \cdot) \# \chi_\Omega = \chi_\Omega \iff \det D\mathbf{F}(t, \mathbf{x}) = 1, \quad (2.14)$$

where $D\mathbf{F}$ is the Jacobian matrix of \mathbf{F} . Hence $\mathbf{F}(t, \cdot)$ is a volume preserving mapping of Ω .

Using (2.13), it is possible to derive an evolution equation for $\nu(t, \mathbf{y})$ in dual space. Namely, for any $\xi \in C_c^1([0, \tau) \times \mathbb{R}^3)$,

$$\int_{[0, \tau) \times \mathbb{R}^3} \left(\frac{\partial}{\partial t} \xi(t, \mathbf{y}) + \mathbf{w}(t, \mathbf{y}) \cdot \nabla \xi(t, \mathbf{y}) \right) \nu(t, \mathbf{y}) d\mathbf{y} dt + \int_{\mathbb{R}^3} \xi(0, \mathbf{y}) \nu(0, \mathbf{y}) d\mathbf{y} = 0, \quad (2.15)$$

where the dual velocity \mathbf{w} is defined (and automatically divergence-free, by its definition) by

$$\mathbf{w}(t, \mathbf{y}) = J(\mathbf{y} - \nabla P^*(t, \mathbf{y})) \implies \nabla \cdot \mathbf{w} = 0. \quad (2.16)$$

with P^* denoting the Legendre transform of P :

$$P^* = \sup_{\mathbf{x} \in \Omega} \{ \mathbf{x} \cdot \mathbf{y} - P(t, \mathbf{x}) \}. \quad (2.17)$$

Equation (2.15) is the weak formulation of the transport equation

$$\frac{\partial}{\partial t} \nu(t, \mathbf{y}) + \nabla \cdot (\mathbf{w}(t, \mathbf{y}) \nu(t, \mathbf{y})) = 0. \quad (2.18)$$

Combining (2.18), (2.16) and the weak formulation of the Monge-Ampère equation (2.14) yields the *semi-geostrophic equations in dual variables*

$$\frac{\partial}{\partial t} \nu(t, \mathbf{y}) + \nabla \cdot (\mathbf{w}(t, \mathbf{y}) \nu(t, \mathbf{y})) = 0, \quad (t, x) \in [0, \tau) \times \mathbb{R}^3, \quad (2.19)$$

$$\mathbf{w}(t, \mathbf{y}) = J(\mathbf{y} - \nabla P^*(t, \mathbf{y})), \quad (t, x) \in [0, \tau) \times \mathbb{R}^3, \quad (2.20)$$

$$\nabla P(t, \cdot) \# \chi_\Omega = \nu(t, \cdot), \quad t \in [0, \tau), \quad (2.21)$$

where J is defined by (2.12) and P^* by (2.17); $\nabla P(t, \cdot)$ is the unique optimal transport map of χ_Ω to $\nu(t, \cdot)$.

Equation (2.21) expresses the energy minimisation requirement, hence it is a precise mathematical formulation of Cullen's principle. Equations (2.19)-(2.21) are supplemented with the initial condition

$$\nu(0, \cdot) = \nu_0(\cdot), \quad \mathbf{y} \in B(0, r) \subset \mathbb{R}^3. \quad (2.22)$$

Note that we require that ν_0 is a given measure with *compact support* contained in some ball $B \subset \mathbb{R}^3$.

The proof of Benamou and Brenier

To prove the existence of weak solutions of the system (2.19)-(2.22), the following strategy was introduced in [7]:

- Given the compactly supported, absolutely continuous measure $\nu(t, \mathbf{y})$ at a given fixed time t , compute the velocity field \mathbf{w} from (2.21) and (2.20).

- In order to advect ν in time using (2.19), the system is discretised in time. Then \mathbf{w} is used to advect ν to the next time step, using the transport equation (2.19). Due to the way in which \mathbf{w} is constructed, we have that $\mathbf{w} \in L_{loc}^\infty([0, \tau] \times \mathbb{R}^3)$ and $\mathbf{w} \in L^\infty([0, \tau]; BV_{loc}(\mathbb{R}^3))$. The measure ν remains compactly supported within a ball whose radius depends on time.
- To solve the transport equation, one must also use a sequence of regularised problems, with Lipschitz continuous velocity field, that approximates \mathbf{w} . For the approximating problems, the transport equation is uniquely solvable. Then, using the stability property of polar factorisation, one can show that these approximate solutions converge to solutions of the system (2.19)-(2.22).

This strategy gives a proof of the main result [7, Theorem 5.1]; our slightly more general statement is taken from [12, Theorem 2.3]:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set such that $\overline{\Omega} \subset B(0, S)$, where $B(0, S)$ is an open ball of radius S centred at the origin. Let $P_0(\mathbf{x})$ be a convex bounded function in $B(0, S)$ satisfying*

$$\nu_0 := \nabla P_0 \# \chi_\Omega \in L^q(\mathbb{R}^3) \quad (2.23)$$

for some $q > 1$. Then, for $\tau > 0$, there exist functions ν on $[0, \tau] \times \mathbb{R}^3$, P on $[0, \tau] \times \Omega$ such that (ν, P) satisfy (2.19)-(2.21) and the initial condition (2.22) in the weak sense. In addition,

(i) ν, P satisfy

$$\nu \in L^\infty([0, \tau]; L^q(\mathbb{R}^3)) \cap C([0, \tau]; L_w^q(\mathbb{R}^3)), \quad (2.24)$$

$$P \in L^\infty([0, \tau]; W^{1, \infty}(\Omega)) \cap C([0, \tau]; W^{1, r}(\Omega)), \quad P(t, \cdot) \text{ is convex in } \Omega;$$

where $r \in [1, \infty)$ and $C([0, \tau]; L_w^q(\mathbb{R}^3))$ is the set of all measurable functions $\mu(t, \mathbf{y})$ on $[0, \tau] \times \mathbb{R}^3$ such that $\mu_{(t)}(\cdot) = \mu(t, \cdot) \in L^q(\mathbb{R}^3)$ for any $t \in [0, \tau]$ and, for any $\{t_k\}_{k=1}^\infty$, $t_* \in [0, \tau]$ satisfying $\lim_{k \rightarrow \infty} t_k = t_*$, we have $\mu_{(t_k)} \rightharpoonup \mu_{(t_*)}$ weakly in $L^q(\mathbb{R}^3)$ (narrowly if $q = \infty$);

(ii) for all $t \in [0, \tau]$, $\text{supp}(\nu(t, \cdot)) \subset B(0, R_0)$, where $R_0 = S(1 + \tau)$;

(iii) $P^* = \sup_{\mathbf{x} \in \Omega} \{\mathbf{x} \cdot \mathbf{y} - P(t, \mathbf{x})\}$ satisfies

$$P^*(t, \cdot) \text{ is convex in } \mathbb{R}^3 \text{ for any } t \in [0, \tau], \quad (2.25)$$

$$P^* \in L_{loc}^\infty([0, \tau] \times \mathbb{R}^3), \quad (2.26)$$

$$\nabla P^* \in L^\infty([0, \tau] \times \mathbb{R}^3; \mathbb{R}^3) \cap C([0, \tau]; L^r(B(0, R); \mathbb{R}^3)),$$

for any $R > 0$ and any $r \in [1, \infty)$. Moreover,

$$\|\nabla P^*(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq S \quad \text{for every } t \in [0, \tau].$$

(iv) $\mathbf{w} \in L_{loc}^\infty([0, \tau] \times \mathbb{R}^3)$, $\mathbf{w} \in L^\infty([0, \tau]; BV_{loc}(\mathbb{R}^3))$.

Remark 2.2. The original result of [7] makes the assumption $q > 3$ in Theorem 2.1. Lopes Filho and Nussenzveig Lopes [23] extended this result to $q > 1$. Loeper [22] extended this result further, proving existence and stability of measure valued solutions. In [19], Faria *et al.* have extended the results of [12] for the incompressible equations to the case of an initial potential density ν_0 in L^1 . Faria has recently done the same for the compressible system as well, [18].

In view of these results, we will include the case $q = 1$ in our main statements below.

The strategy employed to prove Theorem 2.1 can be adapted to prove existence of weak solutions in dual space for the compressible equations [14, 20]. In this paper, we will prove an analogous result for the case of a free boundary condition, using a modification of the original strategy that does not explicitly require the time discretization argument of [7], but relies instead on the theory of Hamiltonian ODEs of [5], summarised in the Appendix. This basic structure of proof was already used in [14].

3 The incompressible free boundary problem

In this section, we study the problem obtained when the rigid boundary condition (1.8) considered in [7] is replaced by a more physically relevant free boundary condition. To model this situation, the equations (1.7) are to be solved in $[0, \tau) \times \Omega_h(t)$, where the domain $\Omega_h(t) \subset \mathbb{R}^3$ is time-dependent and represents the region occupied by the fluid at time t :

$$\Omega_h(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_2, 0 \leq x_3 \leq h(t, x_1, x_2)\}. \quad (3.1)$$

Here $\Omega_2 \subset \mathbb{R}^2$ is a fixed bounded domain with rigid wall boundary conditions, while $h(t, x_1, x_2)$ is unknown and represents the free boundary.

The incompressibility of the flow can be formulated as the requirement that $|\Omega_h(t)|$ remains constant for all $t \in [0, \tau)$, where $|\cdot|$ denotes the three-dimensional Lebesgue measure. In what follows, we normalise the measure so that

$$|\Omega_h(t)| = 1 \quad \text{for all } t < \tau.$$

We denote by $\sigma_h(t, \mathbf{x}) \in \mathcal{P}_{ac}(\mathbb{R}^3)$ the probability measure defined on \mathbb{R}^3 by

$$\sigma_h(t, \mathbf{x}) = \chi_{\Omega_h(t)}(\mathbf{x}), \quad \int_{\mathbb{R}^3} \sigma_h(t, \mathbf{x}) d\mathbf{x} = 1 \quad \forall t < \tau. \quad (3.2)$$

We make no a-priori assumption that $h(t, x_1, x_2)$ is a well defined, single valued function, since in principle the free boundary could develop an overhanging profile. Hence our notation in (3.1) is not well defined. However, we will show that the solution indeed corresponds to a well-defined function, so the abuse of notation in our definition of the domain is ultimately justified.

The flat rigid bottom of the domain is defined by $x_3 = 0$.

The boundary conditions we consider are

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \mathbf{x} \in \partial\Omega_h(t) \setminus \{x_3 = h\}, \quad (3.3)$$

$$\begin{cases} \partial_t h + u_1 \frac{\partial h}{\partial x_1} + u_2 \frac{\partial h}{\partial x_2} = u_3, \\ p(t, x_1, x_2, h(x_1, x_2)) = p_h, \end{cases} \quad \mathbf{x} \in \partial\Omega_h(t) : x_3 = h(t, x_1, x_2), \quad (3.4)$$

where p_h is a prescribed constant; for convenience henceforth we take $p_h = 0$.

In what follows, we first state the results of [13], obtained by taking advantage of the additional assumption of constant density. This assumption reduces the dimensionality of the problem, so that the governing equations are transformed to the shallow water system. We then consider variable density and the incompressible three-dimensional problem, and prove our main result.

3.1 Constant density - the 2-D shallow water equations

When the density is assumed constant, the system (1.7) describing the flow of an incompressible fluid reduces to the *two-dimensional semi-geostrophic shallow water equations*:

$$D_t^{(2)} u_1^g - u_2 + \frac{\partial h}{\partial x_1} = 0, \quad (3.5)$$

$$D_t^{(2)} u_2^g + u_1 + \frac{\partial h}{\partial x_2} = 0, \quad (3.6)$$

$$\frac{\partial h}{\partial t} + \nabla_2 \cdot (h \mathbf{u}_2) = 0, \quad (3.7)$$

$$u_1^g = -\frac{\partial h}{\partial x_2}, \quad u_2^g = \frac{\partial h}{\partial x_1}, \quad (3.8)$$

where $\mathbf{u}_2 = (u_1, u_2)$, $D_t^{(2)} = \partial_t + \mathbf{u}_2 \cdot \nabla$, and all equations are to be solved for $(t, \mathbf{x}) \in [0, \tau) \times \Omega_2$. The system (3.5)-(3.8) is to be considered with the prescribed initial and boundary conditions

$$\mathbf{u}_2 \cdot \mathbf{n} = 0 \quad \text{on } [0, \tau) \times \partial\Omega_2, \quad h(0, \cdot) = h_0(\cdot) \quad \text{in } \Omega_2. \quad (3.9)$$

Note that the evolution of the free boundary $h(t, \mathbf{x})$ is now explicitly part of the system of governing equations, which are posed in the *fixed* domain Ω_2 .

The 2-D geostrophic energy associated with the flow is defined by

$$E_2 = \int_{\Omega_2} \left(\frac{1}{2} ((u_1^g)^2 + (u_2^g)^2) h + \frac{1}{2} h^2 \right) dx_1 dx_2. \quad (3.10)$$

The dual system in Lagrangian coordinates, obtained after passing to the dual coordinates $y_1 = x_1 + u_g^2$, $y_2 = x_2 - u_g^1$, is given by

$$\frac{\partial}{\partial t} \nu(t, \mathbf{y}) + \nabla_2 \cdot (\mathbf{w}(t, \mathbf{y}) \nu(t, \mathbf{y})) = 0, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.11)$$

$$\mathbf{w}(t, \mathbf{y}) = J_2(\mathbf{y} - \nabla_2 P^*(t, \mathbf{y})), \quad \text{in } [0, \tau) \times \mathbb{R}^2, \quad (3.12)$$

$$\nabla_2 P(t, \cdot) \# h(t, \cdot) = \nu(t, \cdot) \quad \text{for any } t \in [0, \tau), \quad (3.13)$$

$$P(t, \mathbf{x}) = h(t, \mathbf{x}) + \frac{1}{2}(x_1^2 + x_2^2), \quad \text{in } [0, \tau) \times \Omega_2, \quad (3.14)$$

$$\nu(0, \mathbf{y}) = \nu_0(\mathbf{y}) \quad \text{given, compactly supported.} \quad (3.15)$$

The main theorem of [13] is summarised below.

Theorem 3.1. *Let $\Omega_2 \subset \mathbb{R}^2$ be an open connected set. Let r be given, $1 \leq r < \infty$. Assume that $\nu_0 \in L^r(\mathbb{R}^2)$, $h_0 \in L^1(\mathbb{R}^2)$ are two probability density functions, such that $\text{support}(\nu_0) \subset B(0, S)$, where $B(0, S)$ is an open ball of radius S centered at the origin. Assume also that the function $P_0(\mathbf{x}) = |\mathbf{x}|^2/2 + h_0(\mathbf{x})$ can be extended to a convex bounded function in \mathbb{R}^2 and that ν_0, h_0 satisfy*

$$\nu_0 = \nabla P_0 \# h_0. \quad (3.16)$$

Then, for $\tau > 0$, there exist functions ν on $[0, \tau) \times \mathbb{R}^2$, P on $[0, \tau) \times \Omega_2$ such that (ν, P) satisfy (3.11)-(3.15) and the initial condition (3.15) in the weak sense. In addition ν, P satisfy the regularity stated in (i)-(iv) of Theorem 2.1.

3.2 Variable density - the incompressible free boundary problem in 3-D

We now consider the incompressible semi-geostrophic system (1.7) in the region $\Omega_h(t)$ given by (3.1), with boundary conditions (3.3)-(3.4).

The energy associated with the flow is the geostrophic energy defined by

$$E = \int_{\Omega_h} \left(\frac{1}{2}((u_1^g)^2 + (u_2^g)^2) + \rho x_3 \right) d\mathbf{x}. \quad (3.17)$$

By a formal but straightforward calculation, it can be shown that, as expected, this energy integral is conserved in time.

Proposition 3.2. *The system (1.7)-(3.34) conserves the energy integral in (3.17).*

Similarly, a formal argument shows that geostrophic and hydrostatic balance can be characterised as a stationary point of the energy in (3.17) with respect to a particular class of variations, supporting the validity of Cullen's stability principle also in this case.

Remark 3.3 (Support of the density $\rho(t, \mathbf{x})$). We can assume that there exists $\delta > 0$ such that the density $\rho(t, \mathbf{x})$ satisfies

$$\delta < \rho(t, \mathbf{x}) < \frac{1}{\delta}, \quad \mathbf{x} \in \Omega_h, \text{ uniformly for } t < \tau. \quad (3.18)$$

This follows from assuming the bound at time $t = 0$ and employing the third of equations (1.7). The full arguments are presented in [20].

Note that the incompressibility condition as expressed by (3.2) and the conservation of energy (3.17) imply that any sufficiently regular $h(t, \cdot)$ which is a solution of the system has to satisfy

$$h(t, \cdot) \in L^1 \cap L^2(\Omega_2), \quad (3.19)$$

at least if it is assumed that $\rho(t = 0)$ satisfies the bound (3.18), and that the energy E is initially bounded.

Indeed,

$$\|h\|_1 = \int_{\Omega_2} h(x_1, x_2) dx_1 dx_2 = \int_{\Omega_2} \int_0^h d\mathbf{x} = \int_{\Omega} d\sigma_h = 1, \quad (3.20)$$

and

$$\begin{aligned} \|h\|_2^2 &= \int_{\Omega_2} h^2(x_1, x_2) dx_1 dx_2 = \int_{\Omega_2} \left[\int_0^h 2x_3 dx_3 \right] dx_1 dx_2 \leq \frac{2}{\delta_\rho} \int_{\Omega_2} \left[\int_0^h \rho x_3 dx_3 \right] dx_1 dx_2 \\ &\leq \frac{2}{\delta_\rho} \int_{\Omega_h} \left(\frac{1}{2}((u_1^g)^2 + (u_2^g)^2) + \rho x_3 \right) d\mathbf{x} = \frac{2}{\delta} E := C_0. \end{aligned} \quad (3.21)$$

We also assume that there exists a constant $H > 0$ such that for every admissible $h(t, \cdot)$,

$$\Omega_h \subset \Omega_2 \times [0, H] := \Omega_H. \quad (3.22)$$

This assumption will be justified by our solution procedure.

3.2.1 Dual formulation

In what follows, we assume that $\Lambda \subset \mathbb{R}^3$ is an open bounded set. Indeed, we assume there exists $R_0 > 0$ such that

$$\Lambda \subset \Lambda_2 \times [-R_0, 0), \quad R_0 > 0, \quad \Lambda_2 \subset \mathbb{R}^2 \text{ bounded.} \quad (3.23)$$

This bound follows from the bound (3.18) on $\rho(\mathbf{x}, t)$, and from the fact that Λ_2 can be assumed to remain bounded. The latter is guaranteed by condition (H1), see section 3.4.

The change of variables to the geostrophic coordinates, for each fixed $h(x_1, x_2)$ describing the domain, is defined in this case by

$$\mathbf{T} : \Omega_h(t) \rightarrow \Lambda, \quad \mathbf{T}(t, \mathbf{x}) = (T_1(t, \mathbf{x}), T_2(t, \mathbf{x}), T_3(t, \mathbf{x})) = (y_1, y_2, y_3),$$

where

$$T_1(\mathbf{x}) = x_1 + u_2^g, \quad T_2(\mathbf{x}) = x_2 - u_1^g, \quad T_3(\mathbf{x}) = -\rho. \quad (3.24)$$

This definition of the mapping \mathbf{T} , and the bound (3.23), imply that, for all $t < \tau$, the geostrophic velocity (u_1^g, u_2^g) remains bounded.

We will denote the inverse of \mathbf{T} by \mathbf{S} (see Theorem 3.9 below);

$$\mathbf{S}(t, \mathbf{y}) = (S_1(t, \mathbf{y}), S_2(t, \mathbf{y}), S_3(t, \mathbf{y})) = \mathbf{T}^{-1}(t, \mathbf{y}), \quad \mathbf{y} \in \Lambda.$$

We show next that, as in the rigid boundary case, the problem can be formulated as an optimal transport problem, whose solution is given by the gradient of a convex function.

We use (3.24) to rewrite the energy in (3.17), at fixed time t , as the following functional in dual space:

$$E[\mathbf{T}, h] = \int_{\Omega_h} \left[\frac{1}{2} \{|x_1 - T_1(\mathbf{x})|^2 + |x_2 - T_2(\mathbf{x})|^2\} - x_3 T_3(\mathbf{x}) \right] d\mathbf{x} \quad (3.25)$$

The following definition is the analogue of Definition 2.1.

Definition 3.1. *Given σ_h as in (3.2), define the potential density $\nu := \mathbf{T} \# \sigma_h \in \mathcal{P}_{ac}(\Lambda)$ associated with the flow described by (1.7)-(3.4) as the push forward of the measure $\sigma_h \in \mathcal{P}_{ac}(\mathbb{R}^3)$ under the map \mathbf{T} given by (3.24).*

Remark 3.4 (Support of the potential density $\nu(\mathbf{x}, t)$). We show below that the potential density $\nu(t, \mathbf{y})$ must satisfy the evolution (3.29). Assuming that at time $t = 0$ the initial potential density ν_0 has compact support in \mathbb{R}^3 , we can deduce that $\text{supp}(\nu)$ is contained in a bounded open set Λ , depending on the time interval length τ , such that $\bar{\Lambda} \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, -\delta]$, for some δ with $0 < \delta < 1$. This follows from a standard fixed-point argument; see, for example, [11, 22].

Define the functional

$$E_\nu(\sigma_h) = \inf_{\mathbf{T}: \mathbf{T} \# \sigma_h = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x}, \quad (3.26)$$

where σ_h is defined in (3.2) and the cost function c is given by

$$c(\mathbf{x}, \mathbf{y}) = \left[\frac{1}{2} \{|x_1 - y_1|^2 + |x_2 - y_2|^2\} - x_3 y_3 \right]. \quad (3.27)$$

Principle 3.1 (Cullen's stability principle). *At each fixed time t , the pair $(\sigma_h, \bar{\mathbf{T}})$ corresponding to a solution of (1.7) with boundary conditions (3.4) minimises the energy (3.25) amongst all pairs (σ_h, \mathbf{T}) where σ_h is given by (3.2) and $\mathbf{T} \# \sigma_h = \nu$. Namely, given $\nu \in \mathcal{P}_{ac}^2(\Lambda)$, a stable solution corresponds to the following minimal value for the energy:*

$$\mathcal{E}(t, \nu) = \inf_{\sigma_h \in \mathcal{H}} E_\nu(\sigma_h) = \inf_{\sigma_h \in \mathcal{H}} \left\{ \inf_{\mathbf{T}: \mathbf{T} \# \sigma_h = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x} \right\}, \quad (3.28)$$

where $\mathcal{H} \subset \mathcal{P}_{ac}(\mathbb{R}^3)$ is an appropriate subset of $\mathcal{P}_{ac}(\mathbb{R}^3)$.

3.2.2 Lagrangian formulation and statement of the main theorem

We formulate the semi-geostrophic system in dual variables in Lagrangian form, in a way entirely analogous to the rigid boundary case. This yields

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{w}) = 0, \quad \text{in } [0, \tau) \times \Lambda, \quad (3.29)$$

$$\mathbf{w}(t, \mathbf{y}) = J(\mathbf{y} - \nabla P^*(t, \mathbf{y})), \quad \text{in } [0, \tau) \times \Lambda, \quad (3.30)$$

$$\nabla P \# \sigma_h = \nu, \quad \nabla P(t, \cdot) \text{ is the unique optimal transport map, and} \quad (3.31)$$

$$\sigma_h \text{ minimises } E_{\nu(t, \cdot)}(\cdot) \text{ over } \mathcal{H}, \quad t \in [0, \tau). \quad (3.32)$$

Here, P^* denotes the Legendre transform of the (convex) function P and \mathcal{H} denotes an appropriate minimisation space, which we define in the next section, see (3.40).

At each fixed time $t < \tau$, the unknowns in this system are the fluid profile $h(t, x_1, x_2)$ and the geopotential $P(t, \mathbf{x})$. We can assume that $h(t, x_1, x_2)$ is a well defined function of $(x_1, x_2) \in \Omega_2$, an assumption justified by the result of Lemma 3.7 below.

Given $P(t, \mathbf{x})$ and $h(x_1, x_2, t)$, it is possible to reconstruct $\nu = \nabla P \# \sigma_h$. Moreover, we show in Proposition 3.54 below that the pressure $p(t, \mathbf{x})$ is obtained from the solution $P(t, \mathbf{x})$ of the system through the relation

$$p(t, \mathbf{x}) = P(t, \mathbf{x}) - \frac{1}{2}(x_1^2 + x_2^2), \quad (t, \mathbf{x}) \in [0, \tau) \times \Omega. \quad (3.33)$$

The system is to be solved, in the weak sense of (2.15), given the following initial conditions

$$h(0, \cdot) = h_0(\cdot) \in W^{1, \infty}(\Omega_2), \quad (x_1, x_2) \in \Omega_2, \quad (3.34)$$

$$\nu(0, \cdot) = \nu_0(\cdot) \text{ compactly supported probability density in } L^r, \quad r \in [1, \infty), \quad (3.35)$$

$$P(0, \mathbf{x}) = P_0(\mathbf{x}) \in W^{1, \infty}(\Omega_{h_0}), \quad (3.36)$$

satisfying the compatibility condition

$$\nabla P_0 \# \sigma_{h_0} = \nu_0. \quad (3.37)$$

It is not difficult to show that, formally, (3.29)-(3.35) yields a stable solution of (1.7), see [20]:

Lemma 3.5. *A sufficiently regular solution of (3.29)-(3.35) yields a solution of (1.7) with initial condition (3.34) and boundary conditions (3.3)-(3.4).*

We can now state the main theorem. The proof is presented in section 3.4.

Theorem 3.6. *Let $1 \leq r < \infty$ and let $\nu_0 \in L^r(\Lambda_0)$ be an initial potential density with support in Λ_0 , where $\Lambda_0 \subset \Lambda_2 \times [-R_0, 0)$ with $R_0 > 0$ and Λ_2 is a bounded open set in \mathbb{R}^2 . Let $c(\cdot, \cdot)$ be given by (3.27).*

Then the system of semi-geostrophic equations in dual variables (3.29)-(3.35) with given conditions (3.34), (3.36) satisfying the compatibility condition (3.37), has a stable weak solution (h, P) such that $\nu = \mathbf{T} \# \sigma_h$, where $\mathbf{T} = \nabla P$, $\sigma_h = \chi_{\Omega_2 \times [0, h]}$, and ν has compact support.

This solution satisfies:

$$(i) \quad \nu(\cdot, \cdot) \in L^\infty([0, \tau]; L^r(\Lambda)), \quad \|\nu(t, \cdot)\|_{L^r(\Lambda)} \leq \|\nu_0(\cdot)\|_{L^r(\Lambda)}, \quad \forall t \in [0, \tau],$$

$$(ii) \quad P(t, \cdot) \in L^\infty([0, \tau]; W^{1, \infty}(\Omega_2)), \quad \|P(t, \cdot)\|_{W^{1, \infty}(\Omega_2)} \leq C = C(\bar{h}, \Lambda, c(\cdot, \cdot)), \\ \forall t \in [0, \tau],$$

$$(iii) \quad h(t, \cdot) \in W^{1, \infty}(\Omega_2), \quad \text{for all } t \in [0, \tau],$$

where Λ is a bounded open domain in \mathbb{R}^3 containing $\text{supp}(\nu)$ for all $t \in [0, \tau]$.

3.3 The minimisation problem (3.28)

In the rest of this section, we fix the time $t \in (0, \tau)$ and often drop the explicit dependence on it from the equations.

Our aim is to prove existence and uniqueness of a minimiser of the functional $E_\nu(h)$ given by (3.28). We do not follow the strategy employed for the proof of the analogous result for the 2-dimensional problem. Indeed, in our case it does not seem straightforward to prove that the energy functional is strictly convex with respect to h . To prove uniqueness of the minimiser, we will consider the Monge-Kantorovich formulation of the problem, following what done in [10] for the more difficult case of a forced axisymmetric flow.

To be able to prove that the minimisation problem (3.28) admits a solution, we first consider what conditions the problem imposes on the minimisation space \mathcal{H} .

We start by showing that, for every fixed value of $t < \tau$, the minimiser has to correspond to a well defined, single-valued function $h(x_1, x_2) \in L^1 \cap L^2(\Omega_2)$.

Lemma 3.7. *The minimiser of (3.28) is given by a σ_h corresponding to $\Omega_h = \Omega_2 \times [0, h(x_1, x_2)]$ with $h(x_1, x_2) \in L^1 \cap L^2(\Omega_2)$.*

Proof. Suppose that \tilde{h} is multi-valued and define the corresponding domain as $\tilde{\Omega}(t)$. Define $\sigma_{\tilde{h}} := \chi_{\tilde{\Omega}(t)}$. Choose a single valued function $h(x_1, x_2)$ such that $|\tilde{\Omega}| = |\Omega_h|$, and transport map \mathbf{R} such that

$$\mathbf{R} \# \sigma_h = \sigma_{\tilde{h}},$$

where $\sigma_h := \chi_{\Omega_h}$. The existence of such a map R is guaranteed by standard optimal transport results. We choose h in such a way that \mathbf{R} can be expressed as $\mathbf{R} = (R_1(\mathbf{x}), R_2(\mathbf{x}), R_3(\mathbf{x})) = (x_1, x_2, x_3 + \varphi(x_1, x_2, x_3))$ where $\varphi(x_1, x_2, x_3) \geq 0$ for all $(x_1, x_2, x_3) \in$

$\Omega_2 \times [0, h]$. Let $\nu \in \mathcal{P}_{ac}(\Lambda)$ and let $\tilde{\mathbf{T}}$ denote the optimal map in the transport of $\sigma_{\tilde{h}}$ to ν with cost function (3.27). Then, since $\tilde{\mathbf{T}} \circ \mathbf{R} \# \sigma_h = \nu$ and \tilde{T}_3 is negative, we have

$$E_\nu(\tilde{\sigma}_h) = \inf_{\mathbf{T} \# \sigma_{\tilde{h}} = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_{\tilde{h}}(\mathbf{x}) d\mathbf{x} \quad (3.38)$$

$$\begin{aligned} &= \int_{\mathbb{R}^3} c(\mathbf{x}, \tilde{\mathbf{T}}(\mathbf{x})) \sigma_{\tilde{h}}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} c(\mathbf{R}(\mathbf{x}), \tilde{\mathbf{T}} \circ \mathbf{R}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x} \\ &\geq \int_{\mathbb{R}^3} c(\mathbf{x}, \tilde{\mathbf{T}} \circ \mathbf{R}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x} \geq \inf_{\mathbf{T} \# \sigma_h = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x} = E_\nu(\sigma_h). \end{aligned} \quad (3.39)$$

Since \tilde{h} is an arbitrary multi-valued function, we conclude that any multi-valued upper boundary will have a corresponding single valued upper boundary which reduces the energy associated with the flow.

The property that $h(x_1, x_2) \in L^1 \cap L^2(\Omega_2)$ follows from (3.20) and (3.21). \square

We now define the subset \mathcal{H} of $\mathcal{P}_{ac}(\mathbb{R}^3)$ on which we minimise the energy.

Definition 3.2. We define the class $\mathcal{H} \subset \mathcal{P}_{ac}(\mathbb{R}^3)$ by

$$\mathcal{H} := \left\{ \sigma_h(t, \cdot) \in \mathcal{P}_{ac}(\mathbb{R}^3) \mid h \in \mathcal{H}_0 \right\}, \quad (3.40)$$

where the $\mathcal{H}_0 \subset L^1 \cap L^2(\Omega_2)$ is given as

$$\mathcal{H}_0 = \left\{ h : [0, \tau) \times \Omega_2 \rightarrow [0, \infty), h(t, \cdot) \in L^1(\Omega_2), \|h(t, \cdot)\|_1 = 1, \int_{\mathbb{R}^3} x_3 d\sigma_h \leq 2C_0 \right\} \quad (3.41)$$

where σ_h is defined in (3.2) and C_0 is as in (3.21).

Our first aim is to show that the functional (3.28) admits a minimizer in this space.

Proposition 3.8. The functional (3.28) admits a minimising pair (σ_h, T) , where $\sigma_h \in \mathcal{H}$, and T is the optimal map T such that $\nu = \mathbf{T} \# \sigma_h$.

Proof. Let

$$\tilde{\mathcal{E}}(\nu) = \inf_{\sigma_h \in \mathcal{H}} \left\{ \inf_{r \in \Gamma(\sigma_h, \nu)} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{y}) r(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \right\}, \quad (3.42)$$

where $\Gamma(\sigma_h, \nu)$ is the set of all bounded measures $\mu \in \mathcal{P}(\Lambda \times \Omega_2 \times [0, \infty))$ with $\pi_1 \# \mu = \nu$, $\pi_2 \# \mu = \sigma_h$ where π_1 is the projection to Λ and π_2 to $\Omega_2 \times [0, \infty)$.

By standard techniques using the narrow topology there exists an optimal $\mu \in \mathcal{P}(\Lambda \times \Omega_2 \times [0, \infty))$ approximated in the narrow topology by a sequence $\mu_n \in \Gamma(\sigma_{h_n}, \nu)$, with $h_n \in \mathcal{H}_0$.

We consider \mathcal{H} endowed with the weak- L^1 -topology. Note that the weak- L^1 -convergence in \mathcal{H} implies narrow convergence and that \mathcal{H} is not closed in the weak- L^1 -convergence. However, by Pettis criterium \mathcal{H} is relative compact and hence w.l.o.g. we can assume that σ_{h_n} converges L^1 -weakly to $\sigma \in L^1(\Omega_2 \times [0, \infty))$ with $0 \leq \sigma \leq 1$ and thus $\pi_2 \# \mu = \sigma d\mathbf{x}$. Therefore μ as an optimal transport plan from ν to $\sigma \in \mathcal{P}_{ac}(\mathbb{R}^3)$ is in fact given by an optimal map T .

Define $h := \int_0^\infty \sigma(x_1, x_2, x_3) dx_3$. It can easily be shown that $h \in \mathcal{H}_0$. We claim that σ is actually equal to σ_h , so that there is a minimiser of the form required. The proof is similar to the proof of lemma 3.7, and is obtained by contradiction by showing that

$$E_\nu(\sigma_h) = \inf_{\mathbf{T}: \mathbf{T} \# \sigma_h = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_h(\mathbf{x}) d\mathbf{x} \leq \inf_{\mathbf{T}: \mathbf{T} \# \sigma = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma(\mathbf{x}) d\mathbf{x} = E_\nu(\sigma).$$

Indeed, consider the transport map \mathbf{R} such that $\mathbf{R}\#\sigma_h = \sigma$, where $\sigma_h := \chi_{\Omega_h}$. The existence of such a map R is guaranteed by standard optimal transport results. It follows from the definition of h and the properties of σ that \mathbf{R} satisfies $R_1(\mathbf{x}) = x_1$, $R_2(\mathbf{x}) = x_2$ and $R_3(\mathbf{x}) \geq x_3$.

Then as in the proof of Lemma 3.7, we deduce that $E_\nu(\sigma_h) \leq E_\nu(\sigma)$, hence the claim. \square

In the remainder of this section, we will show that, at each fixed time t , there exists in fact a unique minimising pair (σ_h, T) of the energy functional (3.28), with $\sigma_h \in \mathcal{H}$, and $T = \nabla P$ for a convex function P .

3.3.1 Kantorovich formulation

We assume that $\nu \in \mathcal{P}_{ac}(\Lambda)$ is a given, compactly supported density, and we consider the cost function $c(\mathbf{x}, \mathbf{y})$ defined by (3.27).

The *Kantorovich dual* of the minimisation problem (3.28) is the problem of maximising the functional

$$J_{(\sigma, \nu)}(f, g) = \int_{\mathbb{R}^3} f(\mathbf{x}) \sigma_h(\mathbf{x}) d\mathbf{x} + \int_{\Lambda} g(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y}, \quad (3.43)$$

$$f \in W^{1, \infty}(\mathbb{R}^3), g \in W^{1, \infty}(\Lambda) : \quad f(\mathbf{x}) + g(\mathbf{y}) \leq c(\mathbf{x}, \mathbf{y}) \text{ for all } (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \Lambda.$$

It can be shown that the solution is unique and the key is in the notion of c -transforms, defined by

$$f^c(\mathbf{y}) = \inf_{\mathbf{x}} (c(\mathbf{x}, \mathbf{y}) - f(\mathbf{x})), \quad g^c(\mathbf{x}) = \inf_{\mathbf{y}} (c(\mathbf{x}, \mathbf{y}) - g(\mathbf{y})).$$

Then (see [15]) there exists a unique point $\bar{\mathbf{x}} \in \mathbb{R}^3$ (respectively $\bar{\mathbf{y}} \in \mathbb{R}^3$), at which the infimum is attained, and which satisfies

$$\nabla f^c(\mathbf{y}) = \nabla_y c(\bar{\mathbf{x}}, \mathbf{y}), \quad \nabla g^c(\mathbf{x}) = \nabla_x c(\mathbf{x}, \bar{\mathbf{y}}). \quad (3.44)$$

To derive explicitly the Kantorovich formulation in the present case, we write the cost $c(\mathbf{x}, \mathbf{y})$ given by (3.27) as

$$\begin{aligned} c(\mathbf{x}, \mathbf{y}) &= \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(y_1^2 + y_2^2) - (x_1 y_1 + x_2 y_2 + x_3 y_3) \\ &= -(\mathbf{x}, \mathbf{y}) + \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(y_1^2 + y_2^2), \end{aligned} \quad (3.45)$$

where (\mathbf{x}, \mathbf{y}) denotes the euclidean inner product in \mathbb{R}^3 .

Then we can write the minimisation problem (3.28) in a relaxed form, and (3.28) can be formulated as the problem of finding $h \in \mathcal{H}_0$ and the optimal plan $\gamma \in \Gamma(\sigma_h, \nu)$ minimising

$$I(\gamma, h) = \int_{\Omega_h \times \Lambda} \left[-(\mathbf{x}, \mathbf{y}) + \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(y_1^2 + y_2^2) \right] d\gamma(\mathbf{x}, \mathbf{y}), \quad (3.46)$$

where

- $\sigma_h \in \mathcal{H}$ with \mathcal{H} given by (3.40);
- ν is a given compactly supported probability measure in \mathbb{R}^3 ;
- Ω_h denotes the support of σ_h , given by (3.1);
- Λ denotes the support of ν , as in (3.23);
- $\Gamma(\sigma_h, \nu)$ denotes the set of probability measure on the product space $\Omega_h \times \Lambda$ which take σ_h and ν as marginals.

The "marginal" condition means that each $\gamma \in \Gamma(\sigma_h, \nu)$ satisfies $\gamma(A \times \Lambda) = \sigma_h(A)$ for every measurable set $A \subset \Omega$, and $\gamma(\Omega \times B) = \nu(B)$ for every measurable set $B \subset \Lambda$.

The Kantorovich maximisation problem can be stated in terms of $P = \frac{1}{2}(x_1^2 + x_2^2) - f$ and $R = \frac{1}{2}(y_1^2 + y_2^2) - g$, where f, g are as in (3.43). The problem is the following:

Problem 3.1 (Kantorovich formulation). *Find $(P(\mathbf{x}), R(\mathbf{y})) \in C(\Omega_H) \times C(\Lambda)$ such that*

$$P(\mathbf{x}) + R(\mathbf{y}) \geq (\mathbf{x}, \mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega_H \times \Lambda, \quad (3.47)$$

and that maximise

$$J(P, R) = \left[\int_{\Lambda} \left(\frac{1}{2}(y_1^2 + y_2^2) - R(\mathbf{y}) \right) d\nu(\mathbf{y}) + \inf_{h \in \mathcal{M}} \int_{\Omega_H} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\sigma_h(\mathbf{x}) \right]. \quad (3.48)$$

where \mathcal{M} is the set of measurable function $h : \Omega_2 \rightarrow [0, \infty)$.

Note that, by construction, for any P, R as above, and any measurable $h : \Omega_2 \rightarrow [0, \infty)$, it holds that

$$J(P, R) \leq I(\gamma, h). \quad (3.49)$$

It follows from the general theory (see appendix) that the solution (P, R) of this maximisation problem is such that P and R are Legendre transforms of each other, namely such that $R = P^*$, $P = R^*$, where

$$P^*(\mathbf{y}) = \sup_{\mathbf{x} \in \Omega_H} ((\mathbf{x}, \mathbf{y}) - P(\mathbf{x})); \quad R^*(\mathbf{x}) = \sup_{\mathbf{y} \in \Lambda} ((\mathbf{x}, \mathbf{y}) - R(\mathbf{y})).$$

In particular this implies that P, R are convex functions, see Lemma 5.1.

3.3.2 Minimisation with respect to the map $\mathbf{T}(\mathbf{x})$

Suppose that $\sigma \in \mathcal{P}_{ac}(\mathbb{R}^3)$ is given. Then the maximisation as stated in Problem 3.1 is the classical optimal transport problem with respect to a quadratic cost between two probability measures absolutely continuous with respect to Lebesgue measure. Details of the general theory are given in appendix, where we state that there exists f that maximises (3.43), and that the optimal transport plan between $\sigma_h(\mathbf{x})$ and $\nu(\mathbf{y})$ is given by $id \times \nabla P$, where $P(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2) - f(\mathbf{x})$, see Lemma 5.1.

Indeed, in this case there exists an optimal transport map \mathbf{T} which solves (3.9). This map is given by $\mathbf{T} = \nabla P$, with P the maximiser in (3.48).

Theorem 3.9. *Assume that $\Lambda \subset \mathbb{R}^3$ is a bounded open set. Let $\sigma_h \in \mathcal{P}_{ac}(\mathbb{R}^3)$, $\nu \in \mathcal{P}_{ac}(\Lambda)$ and $c(\cdot, \cdot)$ be defined by (3.27). Then, there exist maps \mathbf{T} and \mathbf{S} , unique σ_h -a.e. and ν -a.e. respectively, and a convex function P such that*

- (i) $\mathbf{T} = \nabla P$ is optimal in the transport of σ_h to ν with cost $c(\mathbf{x}, \mathbf{y})$,
- (ii) $\mathbf{S} = \nabla P^*$ is optimal in the transport of ν to σ_h with cost $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$, where P^* is defined in (2.17),
- (iii) \mathbf{S} and \mathbf{T} are inverses, i.e. $\mathbf{S} \circ \mathbf{T}(\mathbf{x}) = \mathbf{x}$ for σ_h -a.e. \mathbf{x} and $\mathbf{T} \circ \mathbf{S}(\mathbf{y}) = \mathbf{y}$ for ν -a.e. \mathbf{y} .

We also have the following stability result.

Lemma 3.10. Assume that $\Lambda \subset \mathbb{R}^3$ is a bounded open set. Let $c(\cdot, \cdot)$ be defined by (3.27). Define $E_\nu(\cdot)$ by (3.26). Let $\sigma_{h_n}, \sigma_h \in \mathcal{H}$ and $\nu_n, \nu \in \mathcal{P}_{ac}(\Lambda)$ with σ_{h_n} converging to σ_h in \mathcal{H} and ν_n converging narrowly to ν as $n \rightarrow \infty$. Then $E_{\nu_n}(h_n) \rightarrow E_\nu(h)$ as $n \rightarrow \infty$, i.e.

$$\inf_{\mathbf{T} \# \sigma_{h_n} = \nu_n} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_{h_n} d\mathbf{x} \longrightarrow \inf_{\mathbf{T} \# \sigma_h = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_h d\mathbf{x}, \quad \text{as } n \rightarrow \infty. \quad (3.50)$$

Proof. The proof is standard in optimal transport theory, and similar to the analogous proof in [15], since (??) implies narrow convergence of σ_{h_n} to σ_h . \square

3.3.3 Minimisation with respect to the function $h(x_1, x_2)$

Assume that $P(\mathbf{x})$ is a convex function such that, for fixed (x_1, x_2) , the function $\tilde{P}(x_3) = \frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x})$ is nonzero on a subset $I \subset \mathbb{R}$ of positive Lebesgue measure and satisfies

$$\frac{d\tilde{P}}{dx_3} = -\frac{\partial P}{\partial x_3} \geq 0. \quad (3.51)$$

Given $\nu \in \mathcal{P}_{ac}(\Lambda)$, we aim to prove that there exists a unique measurable function $h(x_1, x_2) : \Omega_2 \rightarrow [0, H]$ which is a minimiser for the second term in (3.48).

Define

$$\Pi_P(x_1, x_2, s) = \int_0^s \left[\frac{1}{2}(x_1^2 + x_2^2) - P(x_1, x_2, x_3) \right] dx_3. \quad (3.52)$$

The function $\Pi_P(x_1, x_2, s)$ admits a minimum in s (by continuity), and it follows from our assumption on $\tilde{P}(x_3)$ that the point s^* where the minimum is attained is unique. Indeed, if $s_* \neq 0$ is such a point, then from the condition $\frac{\partial \Pi_P}{\partial s}(s_*) = 0$ we obtain

$$P(x_1, x_2, s_*) = \frac{1}{2}(x_1^2 + x_2^2), \quad (3.53)$$

i.e. $\tilde{P}(s_*) = 0$ if $s_* \neq 0$ is a minimiser. Integrating by parts the integral in (3.52), we obtain

$$\int_0^s \tilde{P}(x_3) dx_3 = s\tilde{P}(s) - \int_0^s x_3 \frac{d\tilde{P}}{dx_3}.$$

Hence if $s = s_*$ is a minimiser, it follows from (3.53) that the first term on the right hand side vanishes. Hence if two such nonzero points exist, say s_*^1 and s_*^2 , it must be $\int_{s_*^1}^{s_*^2} x_3 \frac{d\tilde{P}}{dx_3} = 0$.

Given our assumption that the integrand is of one sign, this implies $s_*^1 = s_*^2$.

Now suppose $s^* = 0$ is a point of minimum, hence that $\Pi_P(x_1, x_2, s) \geq 0$ for all $s \in [0, H]$. Since the integrand must then be nonnegative, a point s can minimise $\Pi_P(x_1, x_2, s)$ only if (3.53) holds at $s^* = s$.

Hence the minimiser of $\Pi_P(x_1, x_2, s)$ is unique.

Definition 3.3. Define

$$h(x_1, x_2) = s_*$$

where s_* is the unique minimiser of Π_P given by (3.52).

Following the argument presented in [10] for a more difficult situation, one can show that the h given by Definition 3.3 is well defined and that $h(x_1, x_2) \in L^1 \cap L^2(\Omega_2)$.

Remark 3.11. As discussed in the previous section, the optimal transport map between σ_h and the given density ν exists and takes the form $\mathbf{T} = \nabla P_h$, where P_h is convex. Moreover, P_h is related to the physical pressure p by the relation (see equation (3.54) below)

$$P_h = p + \frac{1}{2}(x_1^2 + x_2^2).$$

It follows that

$$\frac{\partial P_h}{\partial x_3} = \frac{\partial p}{\partial x_3} = -\rho \leq 0.$$

Hence P_h satisfies the assumption (3.51).

In addition, since the integrand $x_3\rho(\mathbf{x})$ is nonnegative, the value $s = 0$ cannot be a minimum unless $\rho(\mathbf{x}) = 0$ a.e. (with respect to x_3), a case we exclude, see (3.18).

3.3.4 The minimisation result

In this section, using the results of the minimisation separately in \mathbf{T} and h , we show that there exist a pair of convex functions $(P(\mathbf{x}), R(\mathbf{y}))$ that maximise (3.48), and such that $P(\mathbf{x})$ satisfies the additional requirement to be a monotonic function of the variable x_3 . Then we are able to show that there exists a unique minimiser (γ, h) of (3.46), where $\gamma = Id \times \nabla P$ and $h \in \mathcal{H}_0$, where \mathcal{H}_0 is given in (3.41).

Proposition 3.12. *Let $\nu \in \mathcal{P}_{ac}(\Lambda)$ be given.*

(i) *For all $h \in \mathcal{M}$, there exist a pair $(P(\mathbf{x}), R(\mathbf{y}))$, as in Problem (3.1), that maximise (3.48), and such that $P(\mathbf{x})$ satisfies condition (3.51).*

(ii) *Assume that (γ, h) is an arbitrary pair with $h \in \mathcal{H}_0$ and $\gamma \in \Gamma(\sigma_h, \nu)$, as in (3.46).*

Then $I(\gamma, h) \geq J(P, R)$ for all (P, R) as in (i).

The equality holds if and only if $h(x_1, x_2) \in \mathcal{H}_0$ minimizes $\Pi_P(x_1, x_2, \cdot)$ a.e., and $id \times \nabla P$ is the optimal transport plan of σ_h to ν .

Proof. (i): We sketch the proof, which is based on the analogous proof of part (i) of Proposition 3.4 in [10], but simpler in our case.

The set defined by the conditions in (3.1) is non-empty. Indeed, define

$$c_0 = \sup_{\Omega_H \times \Lambda}(\mathbf{x}, \mathbf{y}), \quad P_0(\mathbf{x}) = c_0, \quad R_0(\mathbf{y}) = 0.$$

Then (P_0, R_0) satisfy all conditions in Problem (3.1).

Now consider a maximising sequence (P_n, R_n) for $J(P, R)$. By the double convexification trick, it can be assumed without loss of generality that this sequence is convex, and it can be shown [10] that it converges uniformly to a pair (P, R) of convex functions satisfying the conditions in Problem (3.1), and in addition such that P satisfies (3.51). The latter is a consequence of the fact that the support Λ of ν has the property (3.23). Then by standard stability results, $J(P, R)$ is a maximum of the functional.

(ii): Let (γ, h_0) be an arbitrary pair with $h_0 \in \mathcal{H}_0$ and $\gamma \in \Gamma(\sigma_{h_0}, \nu)$, and let (P, R) be as in part (i). Then as already observed (see (3.49)) we have using that $P(\mathbf{x}) + R(\mathbf{y}) \geq (\mathbf{x}, \mathbf{y})$

$$J(P, R) = \left[\int_{\Lambda} \left(\frac{1}{2}(y_1^2 + y_2^2) - R(\mathbf{y}) \right) d\nu(\mathbf{y}) + \inf_{h \in \mathcal{M}} \int_{\Omega_H} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\sigma_h(\mathbf{x}) \right] \leq$$

$$\left[\int_{\Lambda} \left(\frac{1}{2}(y_1^2 + y_2^2) - R(\mathbf{y}) \right) d\nu(\mathbf{y}) + \int_{\Omega_{h_0}} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\mathbf{x} \right] \leq I(\gamma, h_0).$$

Equality holds if and only if it holds that

$$\inf_{h \in \mathcal{M}} \int_{\Omega_H} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\sigma_h(\mathbf{x}) = \int_{\Omega_{h_0}} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\mathbf{x}$$

i.e. if $h_0(x_1, x_2)$ is the minimiser of (3.52), and if it holds that

$$\left[\int_{\Lambda} \left(\frac{1}{2}(y_1^2 + y_2^2) - R(\mathbf{y}) \right) d\nu(\mathbf{y}) + \int_{\Omega_{h_0}} \left(\frac{1}{2}(x_1^2 + x_2^2) - P(\mathbf{x}) \right) d\mathbf{x} \right] = I(\gamma, h_0).$$

Using the fact that all measures involved are absolutely continuous with respect to Lebesgue measure, the second condition implies that $\mathbf{y} = \nabla P(\mathbf{x})$ a.e., and hence the map γ is of the form $id \times \nabla P$.

□

Corollary 3.1. *There exists a unique minimiser (γ, h) of the functional $I(\gamma, h)$ given by (3.46), with $h \in \mathcal{H}_0$ and $\gamma \in \Gamma(\sigma_h, \nu)$. In addition, if (P_0, R_0) as in (i) maximises $J(P, R)$, then $J(P_0, R_0) = I(\gamma, h)$ and $(h, T = \nabla P)$ is the unique minimiser of (3.28).*

3.3.5 Properties of the energy minimiser

In this section we use the notation $E_\nu(h) \equiv E_\nu(\sigma_h)$.

Theorem 3.13. *Assume that $\Lambda \subset \mathbb{R}^3$ is a bounded open set and let $\nu \in \mathcal{P}_{ac}(\Lambda)$. Let \mathcal{H} be defined by (3.40). Let \bar{h} correspond to the unique minimiser of $E_\nu(\cdot)$ in \mathcal{H} . Denote by $\bar{\mathbf{T}}$ the optimal map in the transport of $\sigma_{\bar{h}}$ to ν with cost function $c(\mathbf{x}, \mathbf{y})$ defined as in (3.27). Then $\bar{\mathbf{T}} = \nabla P$, where*

$$P = p + \frac{1}{2}(x_1^2 + x_2^2). \quad (3.54)$$

Moreover,

$$p(t, \cdot) \in W^{1,\infty}(\Omega_2 \times [0, \bar{h}]), \quad \bar{h}(t, \cdot) \in W^{1,\infty}(\Omega_2), \quad \text{for all } t \in [0, \tau]. \quad (3.55)$$

Proof. Let ξ be a smooth, compactly supported vector field in $\Omega_2 \times (0, \infty)$ such that $\nabla \cdot \xi = 0$. Consider the one parameter family of measure-preserving diffeomorphisms $\{\mathbf{R}(s, \mathbf{x})\}$ given by

$$\frac{\partial}{\partial s} \mathbf{R}(s, \mathbf{x}) = \xi(\mathbf{R}(s, \mathbf{x})), \quad \mathbf{R}(0, \mathbf{x}) = \mathbf{x}.$$

For $s > 0$, let $\sigma_s \in \mathcal{P}_{ac}(\mathbb{R}^3)$ be given by $\sigma_s := \mathbf{R} \# \sigma_{\bar{h}}$. Since the flow corresponding to ξ is smooth and incompressible, we can assume that for s sufficiently small it transports the initial minimising profile \bar{h} to some perturbed profile \tilde{h}_s , which however may be multivalued. Using Lemma 3.7, we can find a corresponding single-valued $h_s \in L^1 \cap L^2(\Omega_2)$ with $\|h_s\|_1 = 1$, whose corresponding energy is lower than the energy associated with \tilde{h}_s . Hence we can assume in the argument that $\sigma_s = \sigma_{h_s}$.

Since \bar{h} corresponds to the minimiser for E_ν , we have

$$\begin{aligned} 0 \leq E_\nu(h_s) - E_\nu(\bar{h}) &= \inf_{\mathbf{T} \# \sigma_s = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_s(\mathbf{x}) d\mathbf{x} - \inf_{\mathbf{T} \# \sigma_{\bar{h}} = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}(\mathbf{x})) \sigma_{\bar{h}}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}^3} c(\mathbf{x}, \mathbf{T}_s(\mathbf{x})) \sigma_s(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}^3} c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) \sigma_{\bar{h}}(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

where the existence and a.e. uniqueness of optimal maps \mathbf{T}_s and $\bar{\mathbf{T}}$ follow from 3.1. Since $(\bar{\mathbf{T}} \circ \mathbf{R}^{-1})\# \sigma_s = \nu$, and $\mathbf{R}(0, \mathbf{x}) = \mathbf{x}$, we have

$$\begin{aligned} 0 \leq \lim_{s \rightarrow 0} \frac{E_\nu(h_s) - E_\nu(\bar{h})}{s} &\leq \lim_{s \rightarrow 0} \frac{\int_{\mathbb{R}^3} c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{R}^{-1}(s, \mathbf{x}))) \sigma_s(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}^3} c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) \sigma_{\bar{h}}(\mathbf{x}) d\mathbf{x}}{s} \\ &= \lim_{s \rightarrow 0} \frac{\int_{\mathbb{R}^3} [c(\mathbf{R}(s, \mathbf{x}), \bar{\mathbf{T}}(\mathbf{x})) - c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x}))] \sigma_{\bar{h}}(\mathbf{x}) d\mathbf{x}}{s} \\ &= \lim_{s \rightarrow 0} \frac{\int_{\Omega_2} \int_0^{\bar{h}} c(\mathbf{R}(s, \mathbf{x}), \bar{\mathbf{T}}(\mathbf{x})) - c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) d\mathbf{x}}{s} \\ &= \int_{\Omega_2} \int_0^{\bar{h}} \nabla c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) \cdot \xi(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Using the assumption that ξ is an arbitrary vector field in the class chosen, this inequality also holds for $-\xi$. Therefore (using that ξ is divergence free) we can deduce that

$$\int_{\Omega_2} \int_0^{\bar{h}} \nabla \cdot (c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) \xi(\mathbf{x})) d\mathbf{x} = 0.$$

We now want to conclude that $\nabla c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) = -\nabla p(\mathbf{x})$, in the weak sense. Since \bar{h} is not necessarily smooth, we cannot use the Gauss-Green theorem. However we note that Ω_h is a finite perimeter set, i.e. $\sigma_{\bar{h}}$ is of bounded variation. Therefore we can use the generalisation of the divergence theorem due to De Giorgi (see for example [2]) to conclude

$$0 = \int_{\Omega_h} \nabla \cdot (c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) \xi(\mathbf{x})) \sigma_{\bar{h}}(\mathbf{x}) d\mathbf{x} = \int_{\partial \Omega_h^*} ((\xi \cdot \mathbf{n}) c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x}))|_{x_3=\bar{h}}) dx_1 dx_2. \quad (3.56)$$

In this formula, $\partial \Omega_h^*$ denotes the reduced boundary (in the sense of De Giorgi) of Ω_h . The only nonzero boundary terms are the ones arising from the portion of reduced boundary which is a subset of $\{x_3 = \bar{h}(x_1, x_2)\}$. Given the boundary condition $p = 0$ when $x_3 = \bar{h}$, we conclude that the identity (3.56) will hold for arbitrary ξ if the identity $\nabla c(\mathbf{x}, \bar{\mathbf{T}}(\mathbf{x})) = -\nabla p(\mathbf{x})$ holds in the weak sense. Using the fact that $\bar{\mathbf{T}} = \nabla P$ with P convex (see Theorem 3.9), the properties of c -transforms (see (3.44), and the argument for the validity of identity (3.53), this implies that

$$p \in W^{1,\infty}(\Omega), \quad \forall t \in [0, \tau], \quad (3.57)$$

and that

$$P = p + \frac{1}{2}(x_1^2 + x_2^2) \quad a.e. \text{ in } \Omega_2 \times [0, \bar{h}].$$

We now note that since the pressure $p(x_1, x_2, x_3)$ satisfies, for each fixed time $t < \tau$,

$$\begin{cases} \frac{\partial p}{\partial x_3} = -\rho \\ p = p_h \end{cases} \quad x_3 = h(x_1, x_2) \quad (3.58)$$

it is possible to establish a relation between p and h . Indeed, let $\tilde{p}(x_1, x_2) = p(x_1, x_2, h(x_1, x_2))$. By the given boundary conditions, the function \tilde{p} is constant, hence

$$\nabla_2 \tilde{p} = \nabla_2 p + \frac{\partial p}{\partial x_3} \nabla_2 h = 0,$$

where $\nabla_2 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$ denotes the two-dimensional gradient. Using the condition (3.58), we find

$$\nabla_2 p = \rho \nabla_2 h, \quad (x_1, x_2) \in \Omega_2.$$

Hence control of h follows from the above estimates on p and ρ , since $D_t \rho = 0$ implies that ρ is bounded by its initial values. We can therefore assert that the unique solution of the minimisation problem satisfies additionally the property $h \in W^{1,\infty}(\Omega_2)$. \square

Lemma 3.14. *Assume that $\Lambda \subset \mathbb{R}^3$ is a bounded open set. Let $\nu_n, \nu \in \mathcal{P}_{ac}(\Lambda)$ with ν_n converging narrowly to ν as $n \rightarrow \infty$. Let \mathcal{H} be defined by (3.40) and let $E_\nu(\cdot)$ be defined by (3.28). For each n , let $\bar{h}_n \in \mathcal{H}_0$ correspond to the minimiser $\sigma_{\bar{h}_n} \in \mathcal{H}$ of $E_{\nu_n}(\cdot)$ and let $\bar{h} \in \mathcal{H}_0$ correspond to the minimiser $\sigma_{\bar{h}} \in \mathcal{H}$ of $E_\nu(\cdot)$. Then, as $n \rightarrow \infty$, \bar{h}_n converge to \bar{h} in the weak L^1 topology.*

Proof. For each n , $\sigma_{\bar{h}_n}$ minimises $E_{\nu_n}(\cdot)$ over \mathcal{H} so that

$$E_{\nu_n}(\bar{h}_n) \leq E_{\nu_n}(h) \quad \forall h \in \mathcal{H}_0. \quad (3.59)$$

By the compactness of \mathcal{H} , there exists $\tilde{h} \in \mathcal{H}_0$ such that, up to a subsequence that we label \bar{h}_n again, the $\sigma_{\bar{h}_n}$ converge in \mathcal{H} , and hence narrowly, to $\sigma_{\tilde{h}}$.

We now show that \tilde{h} minimises $E_\nu(\cdot)$. From Lemma 3.10, we have that $E_{\nu_n}(\bar{h}_n) \rightarrow E_\nu(\tilde{h})$ as $n \rightarrow \infty$, and that $E_{\nu_n}(\bar{h}) \rightarrow E_\nu(\bar{h})$ as $n \rightarrow \infty$.

Since $\sigma_{\bar{h}}$ minimises (3.26) over \mathcal{H} , we know that $E_\nu(\bar{h}) \leq E_\nu(\tilde{h})$. Assume that $E_\nu(\bar{h}) < E_\nu(\tilde{h})$. Then, for n large enough, $E_{\nu_n}(\bar{h}) < E_{\nu_n}(\bar{h}_n)$. This contradicts (3.59). Thus, we obtain $E_\nu(\tilde{h}) = E_\nu(\bar{h}) \leq E_\nu(h)$ for all $h \in \mathcal{H}$. Hence, we have that \tilde{h} is a minimiser of $E_\nu(\cdot)$. Since the minimiser of $E_\nu(\cdot)$ is unique, we conclude that $\tilde{h} = \bar{h}$. \square

3.4 Dual space existence result

In this section we prove our main result, namely Theorem 3.6.

We will make use of the theory of Hamiltonian ODE of [13], summarised in the Appendix. Here we give the main definition and list the properties of the Hamiltonian that allow us to invoke that theory of Hamiltonian flows in the present context.

The concept of Hamiltonian ODEs is rigorously defined in appendix. In brief, and in the present context, a Hamiltonian flow is the solution $\nu(t) \in \mathcal{P}_{ac}^2$ of the following problem: given an initial probability density $\nu_0 \in \mathcal{P}_{ac}^2$ and a Hamiltonian $H : \mathcal{P}_{ac}^2 \rightarrow \mathbb{R}$, find $\nu(t) \in \mathcal{P}_{ac}^2$ that coincides with ν_0 at time $t = 0$ and satisfying

$$\partial_t \nu(t) + \nabla \cdot (J \partial_0 H(\nu(t)) \nu(t)) = 0.$$

Here $\partial_0 H$ denotes in general the element of minimal L^2 norm in the superdifferential of H (see Appendix). The space \mathcal{P}_{ac}^2 is considered as a metric space with the metric given by the Wasserstein distance W_2 [5].

The strategy of the proof of Theorem 3.6 is to show that when one considers as Hamiltonian the dual energy, it is possible to find a corresponding Hamiltonian flow, and moreover that the velocity $J \partial_0 H(\nu(t))$ coincides with the dual velocity \mathbf{w} .

The three conditions (H1), (H2), (H3) on the Hamiltonian H that guarantee the existence of an Hamiltonian flow are the following:

(H1) *There exist constants $C_0 \in (0, \infty)$, $R_0 \in (0, \infty]$ such that, for all $\nu \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$ with*

$W_2(\nu, \nu_0) < R_0$, we have that the superdifferential is not empty and $\mathbf{v} = \partial_0 H(\nu)$ satisfies $|\mathbf{v}(\mathbf{y})| \leq C_0(1 + |\mathbf{y}|)$ for ν -a.e. $\mathbf{y} \in \mathbb{R}^3$.

(H2) If $\nu, \nu_n \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$, $\sup_n W_2(\nu_n, \nu_0) < R_0$ and $\nu_n \rightarrow \nu$ narrowly, then there exists a subsequence $n(k)$ and functions \mathbf{v}_k, \mathbf{v} such that $\mathbf{v}_k = \partial_0 H(\nu_{n(k)})$ $\nu_{n(k)}$ -a.e., $\mathbf{v} = \partial_0 H(\nu)$ ν -a.e. and $\mathbf{v}_k \rightarrow \mathbf{v}$ a.e. in \mathbb{R}^3 as $k \rightarrow \infty$.

Condition (H1) essentially requires that the velocity's growth is controlled and bounded on every bounded domain, while condition (H2) is a continuity assumption.

To ensure the constancy of H along the solutions of the Hamiltonian system we consider also:

(H3) $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ is proper, upper semi-continuous and λ -concave for some $\lambda \in \mathbb{R}$.

For $\nu \in \mathcal{P}_{ac}^2(\Lambda)$, we define the Hamiltonian $H(\nu)$ as given by dual geostrophic energy:

$$H(\nu) := \mathcal{E}(t, \nu), \quad \mathcal{E}(t, \nu) \text{ given by (3.28)}. \quad (3.60)$$

The main result of [5], Theorem (5.4), states that, if (H1), (H2) hold for $H(\nu)$, then at least for some time there exists an absolutely continuous Hamiltonian flow $\nu_{(t)} \in \mathcal{P}_{ac}(\Lambda)$ satisfying (5.12) such that $t \mapsto \nu_{(t)}$ is Lipschitz, and $\Lambda \subset \mathbb{R}^3$ is a bounded open set. If in addition (H3) holds, then $t \mapsto H(\nu_{(t)})$ is constant.

We begin with showing that the Hamiltonian is superdifferentiable.

Proposition 3.15. *Let $\Lambda \subset \mathbb{R}^3$ be an open bounded set. Let the Hamiltonian $H(\nu)$ on $\mathcal{P}_{ac}^2(\Lambda)$ be defined by (3.60). Then H is superdifferentiable, upper semi-continuous and (-2) -concave.*

Proof. Given $\nu \in \mathcal{P}_{ac}^2(\Lambda)$, denote by \bar{h} the minimiser in (3.28). The existence and a.e. uniqueness of this minimiser follows from Corollary 3.1. For any $\tilde{\nu} \in \mathcal{P}_{ac}^2(\Lambda)$ we have

$$H(\tilde{\nu}) = \inf_{\sigma_h \in \mathcal{H}} \mathcal{E}(\tilde{\nu}, \sigma_h) \leq \mathcal{E}(\tilde{\nu}, \sigma_{\bar{h}}).$$

Let $\mathbf{R}_{\nu}^{\tilde{\nu}}$ be the (unique) optimal transport map from ν to $\tilde{\nu}$ with respect to the usual quadratic cost.

Consider the transport with respect to the cost function $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$ given by (3.27). Let $\mathbf{S}_{\nu}^{\sigma_{\bar{h}}}$ be the optimal map in the transport of ν to $\sigma_{\bar{h}}$ and let $\mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}$ be the optimal map in the transport of $\tilde{\nu}$ to $\sigma_{\bar{h}}$. Therefore, we have

$$\inf_{\mathbf{S} \# \nu = \sigma_{\bar{h}}} \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y}$$

and

$$\inf_{\mathbf{S} \# \tilde{\nu} = \sigma_{\bar{h}}} \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y}.$$

The existence of $\mathbf{S}_{\nu}^{\sigma_{\bar{h}}}$ and $\mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}$ follows from Theorem 3.9. Note that, since $(\mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}} \circ (\mathbf{R}_{\nu}^{\tilde{\nu}})^{-1}) \# \tilde{\nu} = \sigma_{\bar{h}}$ and since $\mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}$ is optimal in the transport of $\tilde{\nu}$ to $\sigma_{\bar{h}}$, we have

$$\int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y} \leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}} \circ (\mathbf{R}_{\nu}^{\tilde{\nu}})^{-1}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y}.$$

It follows that

$$\begin{aligned}
H(\tilde{\nu}) - H(\nu) &\leq \mathcal{E}(\tilde{\nu}, \sigma_{\bar{h}}) - \mathcal{E}(\nu, \sigma_{\bar{h}}) \\
&= \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\
&\leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}} \circ (\mathbf{R}_{\tilde{\nu}}^{\tilde{\nu}})^{-1}(\mathbf{y})) \tilde{\nu}(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\
&= \int_{\Lambda} \tilde{c}(\mathbf{R}_{\tilde{\nu}}^{\tilde{\nu}}(\mathbf{y}), \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\
&= \int_{\Lambda} \left[\tilde{c}(\mathbf{R}_{\tilde{\nu}}^{\tilde{\nu}}(\mathbf{y}), \mathbf{S}_{\tilde{\nu}}^{\sigma_{\bar{h}}}(\mathbf{y})) - \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \right] \nu(\mathbf{y}) d\mathbf{y}, \\
&= \int_{\Lambda} \nabla \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \cdot [\mathbf{R}_{\tilde{\nu}}^{\tilde{\nu}}(\mathbf{y}) - \mathbf{y}] \nu(\mathbf{y}) d\mathbf{y} + o(W_2(\nu, \tilde{\nu})).
\end{aligned} \tag{3.61}$$

Hence, using Definition 5.2, we conclude that $\nabla \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})) \in \partial H(\nu)$. Thus, $\partial H(\nu)$ is non-empty, H is superdifferentiable and we can use Proposition 5.3 to conclude that

$$H \text{ is } (-2) - \text{concave}. \tag{3.62}$$

Also, from the continuity of $\mathcal{E}(\cdot, \cdot)$ (see Lemma 3.10) and the narrow convergence to σ_h as the minimiser of (3.28), we have that

$$H \text{ is upper semi-continuous}. \tag{3.63}$$

From (3.62) and (3.63), we have that (H3) holds. \square

Proposition 3.16. *Let $1 < r < \infty$ and $\nu_0 \in L^r(\Lambda_0)$ be an initial potential density with support in Λ_0 , where Λ_0 is a bounded open set in \mathbb{R}^3 . Let $\Lambda \subset \mathbb{R}^3$ be an open bounded set. Let the Hamiltonian $H = E(t, \nu)$ be defined by (3.28). Then, there exists a Hamiltonian flow $\nu(t) \in \mathcal{P}_{ac}^2(\Lambda)$ and constant $\tau > 0$ such that*

$$\frac{d}{dt} \nu(t) + \nabla \cdot (\tilde{J}(\mathbf{v}(t)) \nu(t)) = 0, \quad \nu(0) = \nu_0, \quad t \in (0, \tau)$$

where $\tilde{J}(\mathbf{v}(t)) = \mathbf{w}$ a.e. in $[0, \tau]$, and for all $t < \tau$, $\text{supp}(\nu(t)) \subset \Lambda$ where Λ is a bounded open set in \mathbb{R}^3 .

Proof. We compute $\partial_0 H(\nu)$ (as defined in Definition 5.2) explicitly to show that the conditions required to apply Theorem 5.4 hold. From the definition of \tilde{J} in (5.11), velocity fields transporting ν will have vanishing components in the y_3 direction so that we need only consider variations of ν in the (y_1, y_2) -directions. Thus, to characterise the elements of $\partial H(\nu)$, we let $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^2)$ and define $\varphi(y_1, y_2, y_3) := \tilde{\varphi}(y_1, y_2)$ for all $\mathbf{y} \in \mathbb{R}^3$. We then set

$$\mathbf{g}_s(\mathbf{y}) = ((g_s)_1(\mathbf{y}), (g_s)_2(\mathbf{y}), (g_s)_3(\mathbf{y})) = \mathbf{y} + s \nabla \varphi(\mathbf{y}).$$

Note that $(g_s)_3(\mathbf{y}) = y_3$ and, for $|s|$ sufficiently small, \mathbf{g}_s is the gradient of a convex function, since $\mathbf{g}_s(\mathbf{y}) = \nabla(\frac{1}{2}\mathbf{y}^2 + s\varphi)$. Define $\nu_s = \mathbf{g}_s \# \nu$. Denote by \bar{h}_s the minimiser in

$$H(\nu_s) = \inf_{\sigma_h \in \mathcal{H}} \mathcal{E}(\nu_s, \sigma_h),$$

and let $\sigma_s := \sigma_{\bar{h}_s}$. The existence and uniqueness of the minimiser \bar{h}_s follows from the minimisation result in Corollary 3.1. Let $\xi \in \partial H(\nu)$. Combining the (-2) -concavity of H and (3.63) with Proposition 5.2, we obtain

$$H(\nu_s) - H(\nu) - \int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + W_2^2(\nu, \nu_s) \leq 0. \quad (3.64)$$

Since, for $|s|$ sufficiently small, \mathbf{g}_s is the gradient of a convex function, we conclude that

$$W_2^2(\nu, \nu_s) = \int_{\Lambda} |\mathbf{y} - \mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} |\mathbf{y} - \mathbf{g}_s(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} = s^2 \int_{\Lambda} |\nabla \varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y}$$

and

$$\int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{g}_s(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} = s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y}.$$

Combining this with (3.64), we therefore obtain

$$\begin{aligned} -s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + s^2 \int_{\Lambda} |\nabla \varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} &\leq H(\nu) - H(\nu_s) \\ &\leq \mathcal{E}(\nu, \sigma_s) - \mathcal{E}(\nu_s, \sigma_s) \\ &= \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y} \\ &\leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s} \circ \mathbf{g}_s(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y} \\ &= \int_{\Lambda} \tilde{c}(\mathbf{g}_s^{-1}(\mathbf{y}), \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y}, \end{aligned} \quad (3.65)$$

since $\mathbf{g}_s \# \nu = \nu_s$. Here $\mathbf{S}_{\nu_s}^{\sigma_s}$ denotes the optimal transport map from ν to σ_s and $\mathbf{S}_{\nu_s}^{\sigma_s}$ denotes the optimal transport map from ν_s to σ_s with respect to the cost function $\tilde{c}(\cdot, \cdot)$. The existence of $\mathbf{S}_{\nu_s}^{\sigma_s}$ and $\mathbf{S}_{\nu_s}^{\sigma_s}$ follows from Theorem 3.9.

Note that

$$\mathbf{g}_s^{-1}(\mathbf{y}) = \mathbf{y} - s \nabla \varphi(\mathbf{y}) + \frac{s^2}{2} \nabla^2 \varphi(\mathbf{y}) \nabla \varphi(\mathbf{y}) + \epsilon(s, \mathbf{y}),$$

where ϵ is a function such that $|\epsilon(s, \mathbf{y})| \leq |s|^3 \|\varphi\|_{C^3(\mathbb{R}^3)}$.

Combining this expression for \mathbf{g}_s^{-1} with (3.65) and using $\frac{\partial}{\partial y_3} \varphi = 0$, we conclude that

$$\begin{aligned} -s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + s^2 \int_{\Lambda} |\nabla \varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} \\ \leq \int_{\Lambda} [\tilde{c}(\mathbf{g}_s^{-1}(\mathbf{y}), \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) - \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))] \nu_s(\mathbf{y}) d\mathbf{y} \\ = \int_{\Lambda} \left[\frac{1}{2} \left\{ |(g_s)_1^{-1}(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |(g_s)_2^{-1}(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} - y_3 (S_{\nu_s}^{\sigma_s})_3(\mathbf{y}) \right. \\ \left. - \frac{1}{2} \left\{ |y_1 - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |y_2 - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} - y_3 (S_{\nu_s}^{\sigma_s})_3(\mathbf{y}) \right] \nu_s(\mathbf{y}) d\mathbf{y} \\ = \int_{\Lambda} \left[\frac{1}{2} \left\{ \left| y_1 - s \frac{\partial}{\partial y_1} \varphi(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y}) \right|^2 + \left| y_2 - s \frac{\partial}{\partial y_2} \varphi(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y}) \right|^2 \right\} \right. \\ \left. - \frac{1}{2} \left\{ |y_1 - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |y_2 - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} \right] \nu_s(\mathbf{y}) d\mathbf{y} + o(s) \\ = s \int_{\Lambda} (\mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}) - \mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu_s(\mathbf{y}) d\mathbf{y} + o(s). \end{aligned}$$

By the definitions of \mathbf{g}_s and ν_s , we have that $\nu_s \rightarrow \nu$ in $\mathcal{P}_{ac}(\Lambda)$ as $s \rightarrow \infty$. Then, by Lemma 3.14, we have that $\sigma_s \rightarrow \sigma_{\bar{h}}$ in \mathcal{H} as $s \rightarrow 0$, where $\sigma_{\bar{h}}$ denotes the unique minimiser in (3.28). Hence, dividing both sides first by $s > 0$, then by $s < 0$ and letting $|s| \rightarrow 0$, we use the natural stability of optimal maps to obtain

$$-\int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} = \rho_0 \int_{\Lambda} (\mathbf{S}_{\nu_{\bar{h}}}^{\sigma_{\bar{h}}}(\mathbf{y}) - \mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y}.$$

Thus, we have that $\tilde{J}(\pi_{\nu} \xi(\mathbf{y})) = \tilde{J}(\rho_0 (\mathbf{y} - \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y})))$, where $\pi_{\nu} : L^2(\nu; \Lambda) \rightarrow T_{\nu} \mathcal{P}_{ac}^2(\Lambda)$ denotes the canonical orthogonal projection, with the tangent space defined by (5.9). The minimality of the norm of $\partial_0 H$ then gives

$$\tilde{J}(\partial_0 H(\nu)) = \tilde{J}(\rho_0 (\mathbf{y} - \mathbf{S}_{\nu}^{\sigma_{\bar{h}}}(\mathbf{y}))) = \mathbf{w}(\mathbf{y}), \quad (3.66)$$

where \mathbf{w} is defined as in (3.30).

We can now check directly that conditions (H1) and (H2) hold. Condition (H1) follows from the Theorem 3.9, which tells us that the optimal map $\mathbf{S}_{\nu}^{\sigma_{\bar{h}}}$ is the gradient of a convex function. Condition (H2) follows from the stability of optimal maps (see [6]). Hence we may apply the result of Theorem 5.4 to conclude that there exists a Hamiltonian flow $\nu_{(t)}$ such that

$$\frac{d}{dt} \nu_{(t)} + \nabla \cdot (\tilde{J}(\mathbf{v}_{(t)}) \nu_{(t)}) = 0, \quad \nu_{(0)} = \nu_0, \quad t \in (0, \tau)$$

where $\tilde{J}(\mathbf{v}_{(t)}) = \tilde{J}(\partial_0 H(\nu_{(t)}))$ for a.e. $t \in [0, \tau]$. By (3.66), this then completes the proof that the dual space continuity equation (3.29), with velocity field defined as in (3.30), is satisfied. In addition, from (H3) and the definition of \tilde{J} , the energy associated with the flow is conserved.

The boundedness of the support of $\nu_{(t)}$ also follows Theorem (5.4). \square

Proof of the main Theorem 3.6

From the definition of \mathbf{w} in Proposition 3.16, we have that (\bar{h}, \mathbf{T}) is a stable solution of (3.29)-(3.35), where $\mathbf{T} = \nabla P$ (see Theorem 3.9). Theorem 3.6 (i) follows from (5.14), (5.15); Theorem 3.6 (ii) follows from Theorem 3.13; Theorem 3.6. Note also that by the definition of \mathbf{w} in terms of the optimal map \mathbf{T}^{-1} which, by Theorem 3.9, is the gradient of a convex function. \square

In summary, by rewriting the incompressible semi-geostrophic equations in an appropriate set of geostrophic coordinates and reformulating the problem as a coupled optimal transport/continuity problem, we have been able to show the existence of stable weak solutions in dual space.

4 The free boundary problem for the compressible semi-geostrophic system

In this section, we generalise the proof of the previous section to hold for the compressible system (1.1)-(1.5). The boundary conditions are (3.3)-(3.4), as before.

The results on the existence of dual solutions for the compressible free boundary problem rely on formulating the equations in the so-called pressure coordinates. In this form, the problem in dual coordinates is formally identical to the one of the previous section for the incompressible case, but formulated with respect to a different cost.

We will formulate the problem, and state the main result. All details can be found in [20].

4.1 Formulation in pressure coordinates

We consider the fully compressible system (1.1)-(1.5). The equations are to be solved in the variable domain defined by (3.1).

The *geostrophic energy* associated with the flow is defined as

$$E(t) = \int_{\Omega} \left[\frac{1}{2} |\mathbf{u}^g|^2(t, \mathbf{x}) + \phi(\mathbf{x}) + c_v \theta(t, \mathbf{x}) \left(\frac{p(t, \mathbf{x})}{p_{\text{ref}}} \right)^{\frac{\kappa-1}{\kappa}} \right] \rho(t, \mathbf{x}) d\mathbf{x}. \quad (4.1)$$

It follows from the hydrostatic balance approximation $\frac{\partial p}{\partial x_3} = -\rho$ that $\frac{\partial p}{\partial x_3}$ is always negative. Hence, the change of variables $x_3 = x_3(p)$ is well-defined and we can express any function ψ of (t, x_1, x_2, x_3) in terms of (t, x_1, x_2, p) by considering x_3 as a dependent variable. In these new coordinates, the compressible semi-geostrophic equations take the form

$$\begin{cases} \frac{D_p u_1^g}{Dt} - u_2 + \frac{\partial \phi}{\partial x_1} = 0, \\ \frac{D_p u_2^g}{Dt} + u_1 + \frac{\partial \phi}{\partial x_2} = 0, \\ \nabla_p \cdot \mathbf{u}_p = 0, \\ \frac{D_p \theta}{Dt} = 0, \\ u_1^g = -\frac{\partial \phi}{\partial x_2}, \quad u_2^g = \frac{\partial \phi}{\partial x_1}, \\ \frac{\partial \phi}{\partial p} = -\frac{R\theta p^{\kappa-1}}{p_{\text{ref}}^{\kappa}}, \end{cases} \quad (t, \mathbf{x}_p) \in [0, \tau) \times \Omega_p(t), \quad (4.2)$$

where $\mathbf{u}_p = (u_1, u_2, \omega)$ denotes the velocity in pressure coordinates,

$$\Omega_p(t) = \{(x_1, x_2, p) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_2, p_h \leq p \leq p_s(t, x_1, x_2)\},$$

and p_s is unknown, while $p_h \geq 0$ is the constant pressure at the fixed boundary. The boundary conditions read

$$\mathbf{u}_p \cdot \mathbf{n} = 0 \quad (x_1, x_2, p) \in \partial\Omega_p(t) \setminus \{p = p_s\}, \quad (4.3)$$

$$x_3 = 0, \quad \frac{D_p p_s}{Dt} = \omega, \quad \text{for } p = p_s. \quad (4.4)$$

We are also given the initial condition

$$p_s(0, \cdot) = (p_s)_0(\cdot) \in C(\Omega_2) \cap W^{1,\infty}(\Omega_2). \quad (4.5)$$

The energy associated with the flow, in \mathbf{x}_p coordinates, takes the form

$$\begin{aligned} E_p &= \int_{\Omega_p(t)} \left[\frac{1}{2} ((u_1^g)^2 + (u_2^g)^2) + \frac{c_p \theta p^{\kappa}}{p_{\text{ref}}^{\kappa}} \right] d\mathbf{x}_p \\ &= \int_{\Omega_2} \int_{p_h}^{p_s(t, x_1, x_2)} \left[\frac{1}{2} ((u_1^g)^2 + (u_2^g)^2) + \frac{c_p \theta p^{\kappa}}{p_{\text{ref}}^{\kappa}} \right] dx_1 dx_2 dp. \end{aligned} \quad (4.6)$$

4.1.1 Formulation in dual coordinates

Assume that $\Lambda \subset \mathbb{R}^3$ is an open bounded set. As in Section 3.2.1, we perform a change of variables to geostrophic coordinates $\mathbf{y} \in \Lambda$. This change of variables is now given by

$$y_1 = x_1 + u_2^g, \quad y_2 = x_2 - u_1^g, \quad y_3 = -\frac{c_p \theta}{p_{\text{ref}}^{\kappa}}. \quad (4.7)$$

We will denote by \mathbf{T} the change of variables from physical to geostrophic coordinates, i.e.

$$\mathbf{T}(t, \mathbf{x}_p) = (T_1(t, \mathbf{x}_p), T_2(t, \mathbf{x}_p), T_3(t, \mathbf{x}_p)) = (y_1, y_2, y_3).$$

We use (4.7) to rewrite the energy in (4.6) as

$$\begin{aligned} E_p &= \int_{\Omega_2} \int_{p_h}^{\bar{p}_s} \left[\frac{1}{2} \{ |x_1 - y_1|^2 + |x_2 - y_2|^2 \} - p^\kappa y_3 \right] d\mathbf{x}_p \\ &= \int_{\mathbb{R}^3} \left[\frac{1}{2} \{ |x_1 - T_1(\mathbf{x}_p)|^2 + |x_2 - T_2(\mathbf{x}_p)|^2 \} - p^\kappa T_3(\mathbf{x}_p) \right] \sigma_{\bar{h}} d\mathbf{x}_p. \end{aligned} \quad (4.8)$$

where $\sigma_{\bar{h}} := \chi_{\Omega_2 \times [p_h, \bar{p}_s]}$.
Define

$$H_p := \left\{ p_s : [0, \tau) \times \Omega_2 \rightarrow (0, \infty), p_s \in W^{1,\infty}(\Omega_2), \|p_s\|_1 = 1, \int_{\mathbb{R}^3} x_3 d\sigma_{p_s} \leq M_0 \right\}, \quad (4.9)$$

and

$$\mathcal{H}_p := \{ \sigma_{p_s}(t, \cdot) \in \mathcal{P}_{ac}(\mathbb{R}^3) \mid p_s \in H_p \}. \quad (4.10)$$

This space is the analogue of (3.40), but with respect to pressure coordinates. As in Section 3.3, it can be shown that the space \mathcal{H}_p is compact in $\mathcal{P}_{ac}(\mathbb{R}^3)$. One can again appeal to Lemma 3.7 to show that, in order for p_s to correspond to an energy minimiser, p_s must be a well-defined single valued function.

Define the potential density $\nu := \mathbf{T} \# \sigma_{\bar{h}}$ as the push forward of the measure $\sigma_{\bar{h}}$ under the map \mathbf{T} .

Then, given $\nu \in \mathcal{P}_{ac}(\Lambda)$, we define for any $p_s \in \mathcal{H}_p$, the functional

$$E_\nu(p_s) = \inf_{\mathbf{T} \# \sigma = \nu} \int_{\mathbb{R}^3} c(\mathbf{x}_p, \bar{\mathbf{T}}(\mathbf{x}_p)) \sigma(\mathbf{x}_p) d\mathbf{x}_p = \inf_{\mathbf{T} \# \sigma = \nu} \int_{\Omega_2} \int_{p_h}^{p_s} c(\mathbf{x}_p, \bar{\mathbf{T}}(\mathbf{x}_p)) d\mathbf{x}_p, \quad (4.11)$$

where

$$c(\mathbf{x}_p, \mathbf{y}) = \left[\frac{1}{2} \{ |x_1 - y_1|^2 + |x_2 - y_2|^2 \} - p^\kappa y_3 \right]. \quad (4.12)$$

The analogue of Principle 2.1 (Cullen's stability principle) now holds for E_p . Hence the dual space semi-geostrophic system then takes the form

$$\begin{aligned} \frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{w}) &= 0, \\ \mathbf{w}(t, \mathbf{y}) &= f_{\text{cor}} J(\mathbf{y} - \mathbf{T}^{-1}(t, \mathbf{y})), \\ \mathbf{T}(t, \cdot) &\text{ is the unique optimal map from } \chi_{\Omega_2 \times [p_h, \bar{p}_s]} \text{ to } \nu \text{ with cost (4.12),} \\ \bar{p}_s(t, \cdot) &\text{ minimises } E_{\nu(t, \cdot)}(\cdot) \text{ over } \mathcal{H}_p, \end{aligned} \quad (4.13)$$

where

$$\nu(0, \cdot) = \nu_0(\cdot) \text{ compactly supported probability density in } L^r, r \in (1, \infty). \quad (4.14)$$

4.1.2 The main existence theorem

Theorem 4.1. *Let $1 \leq r < \infty$ and let $\nu_0 \in L^r(\Lambda_0)$ be an initial potential density with support in Λ_0 , where Λ_0 is a bounded open set in \mathbb{R}^3 . Let $c(\cdot, \cdot)$ be given by (4.12). Then the system of semi-geostrophic equations in dual variables (4.13) has a stable weak solution (\bar{p}_s, \mathbf{T}) with $\bar{p}_s(t, \cdot) \in W^{1,\infty}(B(0, S))$ for some $S > 0$.*

For $\nu = \mathbf{T} \# \sigma_{\bar{h}}$, where $\sigma_{\bar{h}} = \chi_{\Omega_2 \times [p_h, \bar{p}_s]}$, and \mathbf{w} as in (3.30), this solution satisfies

- (i) $\nu(\cdot, \cdot) \in L^r((0, \tau) \times \Lambda), \quad \|\nu(t, \cdot)\|_{L^r(\Lambda)} \leq \|\nu_0(\cdot)\|_{L^r(\Lambda)}, \quad \forall t \in [0, \tau],$
- (ii) $\phi(t, \cdot) \in W^{1,\infty}(\Omega_2 \times [p_h, \bar{p}_s]), \quad \|\phi(t, \cdot)\|_{W^{1,\infty}(\Omega_2 \times [p_h, \bar{p}_s])} \leq C = C(\Omega_2 \times [p_h, \bar{p}_s], \Lambda, c(\cdot, \cdot)),$
 $\forall t \in [0, \tau],$
- (iii) $\|\mathbf{w}(t, \cdot)\|_{L^\infty(\Lambda)} \leq C = C(\Omega_2 \times [p_h, \bar{p}_s], \Lambda), \quad \forall t \in [0, \tau],$

where Λ is a bounded open domain in \mathbb{R}^3 containing $\text{supp}(\nu)$.

Conclusions

We have given a rigorous proof of the existence of dual space solutions for the semi-geostrophic system posed in a domain with variable height, in three dimension. The proof builds on the previous techniques introduced by Benamou-Brenier and Cullen, Gangbo and Maroofi, and it makes use of the general theory of Hamiltonian flows of Ambrosio-Gangbo. The proof is given in detail for the incompressible case, where we outline all the difficulties that need to be overcome. A similar proof then holds also for the compressible set of equations, whose formal structure in pressure coordinates is analogous to the structure of the incompressible ones.

We expect that it should be possible to use arguments similar to the ones in [12] to extend the validity of the main result presented here to the case of weak Lagrangian solutions in physical space.

5 Acknowledgements

DKG gratefully acknowledges the support of an EPSRC-CASE studentship sponsored by the MET Office.

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Appendix

Useful Conventions, Notation and Definitions

We list here notation and conventions used in the paper.

Physical variables and constants (5.1)

- (i) Ω denotes an open bounded convex set in \mathbb{R}^3 , representing the physical domain containing the fluid; $\tau > 0$ is a fixed positive constant; all functions in physical coordinates are defined for $(t, \mathbf{x}) \in [0, \tau) \times \Omega$;
- (ii) $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ represents the full velocity of the fluid;
- (iii) $\mathbf{u}^g(t, \mathbf{x}) = (u_1^g(t, \mathbf{x}), u_2^g(t, \mathbf{x}), 0)$ represents the (two-dimensional) geostrophic velocity;
- (iv) $p(t, \mathbf{x})$ represents the pressure;
- (v) $\rho(t, \mathbf{x})$ represents the density;
- (vi) $\theta(t, \mathbf{x})$ represents the potential temperature. Given its physical meaning, we assume $\theta(t, \mathbf{x})$ to be strictly positive and bounded;
- (vii) ϕ is the prescribed geopotential. We assume that $\phi = g_{\text{grav}} x_3$, where g_{grav} denotes the constant acceleration due to gravity;
- (viii) f_{cor} denotes the Coriolis parameter, which we assume to be constant; in all that follows, we assume $f_{\text{cor}} = 1$;
- (ix) p_{ref} is the reference value of the pressure; R represents the gas constant.

Notations and other conventions (5.2)

- The Lebesgue measure of any set A in \mathbb{R}^3 will be denoted by $|A|$.
- (a) Given an open set A in \mathbb{R}^3 , we will denote by
 - χ_A - the characteristic function of A ;
 - $P_{ac}(A)$ - the set of probability measures in \mathbb{R}^3 with supports contained in A , absolutely continuous with respect to Lebesgue measure.
- Given some function $H : A \rightarrow (-\infty, +\infty]$, we denote by $D(H)$ the set of all $a \in A$ such that $H(a) < +\infty$. We say that H is proper if $D(H) \neq \emptyset$.
- (b) Unless otherwise specified, measurable means Lebesgue measurable and *a.e.* means Lebesgue-*a.e.*
- (c) D_t denotes the Lagrangian derivative, defined as $D_t = \partial_t + \mathbf{u} \cdot \nabla$, where \mathbf{u} denotes the full velocity of the flow as in (ii).
- (d) For convenience, we will sometimes use the notation $F_{(t)}(\cdot) = F(t, \cdot)$ to denote the map F evaluated at fixed time t .
- (e) $W^{1,\infty}$ denotes the usual Sobolev space of essentially bounded functions with first weak derivative in L^∞ .

Optimal transport

For all general definitions regarding probability measures, classical existence of the solution of the optimal transport problem with respect to a quadratic cost, and the Wasserstein metric, we refer to [6]. In this section we only discuss results we need to generalise for the purpose of the present paper.

The optimal transport results of [14, 15], that provide the basis for the present results, utilise the Kantorovich dual problem and many useful properties of its c -transform solutions. We start by defining the c -transforms of functions $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f^c(\mathbf{y}) := \inf_{\mathbf{x} \in \mathbb{R}^3} \{c(\mathbf{x}, \mathbf{y}) - f(\mathbf{x})\} \quad (5.3)$$

and

$$g^c(\mathbf{x}) := \inf_{\mathbf{y} \in \mathbb{R}^3} \{c(\mathbf{x}, \mathbf{y}) - g(\mathbf{y})\}, \quad (5.4)$$

for some cost function $c(\cdot, \cdot)$. We say that f is c -concave if and only if $f = g^c$ for some function g .

For the cost function we will consider here, given by (3.27), we have the following useful characterisation of c -transforms which allows us to conclude that the optimal map in Theorem 3.9 is indeed the gradient of a convex function.

Lemma 5.1. *Let Λ be a bounded open set in \mathbb{R}^3 and let $c(\mathbf{x}, \mathbf{y})$ be given by (3.27). Then f is a c -concave function from \mathbb{R}^3 into \mathbb{R} if and only if $\mathbf{x} \mapsto \overline{P}(\mathbf{x})$, defined by*

$$\overline{P}(\mathbf{x}) := -f(\mathbf{x}) + \frac{1}{2}(x_1^2 + x_2^2) \quad (5.5)$$

is convex.

Proof. We know that f is c -concave if and only if $f = g^c$ for some function g defined on a bounded set $\overline{\Lambda} \subset \mathbb{R}^3$ into \mathbb{R} , i.e.

$$f(\mathbf{x}) = \inf_{\mathbf{y} \in \overline{\Lambda}} \{c(\mathbf{x}, \mathbf{y}) - g(\mathbf{y})\} \quad (5.6)$$

$$\begin{aligned} &= \inf_{\mathbf{y} \in \overline{\Lambda}} \left\{ \left[\frac{1}{2} \{|x_1 - y_1|^2 + |x_2 - y_2|^2\} - x_3 y_3 \right] - g(\mathbf{y}) \right\} \\ &= \inf_{\mathbf{y} \in \overline{\Lambda}} \left\{ \left[\frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(y_1^2 + y_2^2) - \mathbf{x} \cdot \mathbf{y} \right] - g(\mathbf{y}) \right\}, \end{aligned} \quad (5.7)$$

which holds if and only if

$$f(\mathbf{x}) - \frac{1}{2}(x_1^2 + x_2^2) = \inf_{\mathbf{y} \in \overline{\Lambda}} \left\{ -\mathbf{x} \cdot \mathbf{y} - g_0(\mathbf{y}) + \frac{1}{2}(y_1^2 + y_2^2) \right\},$$

i.e.

$$\frac{1}{2}(x_1^2 + x_2^2) - f(\mathbf{x}) = \sup_{\mathbf{y} \in \overline{\Lambda}} \left\{ \mathbf{x} \cdot \mathbf{y} - \left(\frac{1}{2}(y_1^2 + y_2^2) - g(\mathbf{y}) \right) \right\}.$$

Defining

$$\overline{R}(\mathbf{y}) := -g(\mathbf{y}) + \frac{1}{2}(y_1^2 + y_2^2) \quad (5.8)$$

we see that f is c -concave if and only if \overline{P} is the Legendre transform of some function \overline{R} , i.e. if and only if \overline{P} is convex. \square

Hamiltonian Flows

The semi-geostrophic problem can be formulated as coupling an energy minimisation problem with a transport equation, with certain specific regularity properties. The original proof in [7] used a time-discretisation argument to prove the solution of the relevant transport equation exists. However, using a more recent result of Ambrosio and Gangbo [5] of Hamiltonian ODEs in the Wasserstein space of probability measures, it can be shown that the solution of the energy minimisation problem yields a solution of the associated transport equation, through the fact that the velocity field is precisely realised as the superdifferential of the energy.

Here, we summarise the results of [5] that we use in our main proof. While these results may appear technical, they essentially state that if the Hamiltonian of the system, i.e. the energy in dual space, satisfies certain conditions, then the Hamiltonian flow whose velocity field is given by the superdifferential of the energy exists.

In what follows, we let μ, ν, σ be arbitrary measures belonging to $\mathcal{P}^2(\mathbb{R}^d)$, the space of probability measures on \mathbb{R}^d with finite second order moments. We define the tangent space to $\mathcal{P}^2(\mathbb{R}^d)$ at ν as

$$T_{\nu(t)}\mathcal{P}^2(\mathbb{R}^d) = \overline{\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\nu; \mathbb{R}^d)}. \quad (5.9)$$

Definition 5.1. *Given some function $H : A \rightarrow (-\infty, \infty]$, we denote by $D(H)$ the set of all $a \in A$ such that $H(a) < \infty$. We say that H is proper if $D(H) \neq \emptyset$.*

The space $\mathcal{P}^2(\mathbb{R}^d)$, equipped with the Wasserstein metric W_2 is a complete and separable space, but is not locally compact since narrow convergence of measures does not necessarily imply convergence of second order moments. Following [1], we generalise the notions of differentiability and convexity to the metric space $(\mathcal{P}^2(\mathbb{R}^d), W_2)$. In what follows, we deal with concave rather than convex functions. This is due to the way in which we define our Hamiltonian H to represent the minimal energy associated with the flow. Hence, in what follows we replace all definitions and results involving subdifferentiability and λ -convexity given in [1] with results involving superdifferentiability and λ -concavity. We also restrict our attention only to measures which are absolutely continuous with respect to Lebesgue measure.

Definition 5.2. *Let $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ be a proper, upper semi-continuous function and let $\nu \in D(H)$. We say that $\mathbf{v} \in L^2(\nu; \mathbb{R}^3)$ belongs to the Fréchet superdifferential $\partial H(\nu)$ if*

$$H(\tilde{\nu}) \leq H(\nu) + \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y}) \cdot (\mathbf{R}_\nu^{\tilde{\nu}}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + o(W_2(\nu, \tilde{\nu})) \quad \text{as } \tilde{\nu} \rightarrow \nu, \quad (5.10)$$

where $\mathbf{R}_\nu^{\tilde{\nu}}$ is the optimal map in the transport of ν to $\tilde{\nu}$. We denote by $\partial_0 H(\nu)$ the element of $\partial H(\nu)$ of minimal $L^2(\nu; \mathbb{R}^3)$ -norm.

Note that, by the minimality of its norm, $\partial_0 H(\nu)$ belongs to $\partial H(\nu) \cap T_\nu \mathcal{P}_{ac}^2(\mathbb{R}^3)$.

Definition 5.3. *Let $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ be proper and let $\lambda \in \mathbb{R}$. We say that H is λ -concave if, for every $\tilde{\nu}_0, \tilde{\nu}_1 \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$ denoting by \mathbf{T} optimal map in the transport of $\tilde{\nu}_0$ to $\tilde{\nu}_1$, we have*

$$H(\nu_{(t)}) \geq (1-t)H(\tilde{\nu}_0) + tH(\tilde{\nu}_1) - \frac{\lambda}{2}t(1-t)W_2^2(\tilde{\nu}_0, \tilde{\nu}_1)$$

for all $t \in [0, 1]$, where $\nu_{(t)} = \mathbf{T}_\#[(1-t)\tilde{\nu}_0 + t\tilde{\nu}_1]$.

Proposition 5.2. *Let $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ be upper semi-continuous and λ -concave for some $\lambda \in \mathbb{R}$ and let $\nu \in D(H)$. Then, the following condition is equivalent to $\mathbf{v} \in \partial H(\nu)$:*

$$H(\tilde{\nu}) \leq H(\nu) + \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y}) \cdot (\mathbf{R}_{\nu}^{\tilde{\nu}}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + \frac{\lambda}{2} W_2^2(\nu, \tilde{\nu}) \quad \text{for all } \tilde{\nu} \in \mathcal{P}_{ac}^2(\mathbb{R}^3),$$

where $\mathbf{R}_{\nu}^{\tilde{\nu}}$ is the optimal map in the transport of ν to $\tilde{\nu}$.

We have also the following useful result from [25, Proposition 10.12] which provides us with a link between superdifferentiability and λ -concavity in the specific case when $\lambda = -2$ (i.e. semi-concavity):

Proposition 5.3. *Let $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ be a proper, upper semi-continuous function. If H is locally superdifferentiable, then H is also locally (-2) -concave.*

We can now define Hamiltonian ODEs as follows:

Definition 5.4. *Let $H : \mathcal{P}_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, \infty]$ be a proper, upper semi-continuous function. Define the linear transformation $\tilde{J} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by*

$$\tilde{J}(v_1(\mathbf{y}), v_2(\mathbf{y}), v_3(\mathbf{y})) = y_3(-v_2(\mathbf{y}), v_1(\mathbf{y}), 0), \quad (5.11)$$

for all $\mathbf{v}(\mathbf{y}) \in \mathbb{R}^3$. We say that an absolutely continuous curve $\nu_{(t)} : [0, \tau] \rightarrow D(H)$ is a Hamiltonian ODE relative to H , starting from $\nu_0 \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$, if there exists $\mathbf{v}_{(t)} \in L^2(\nu_{(t)}; \mathbb{R}^3)$ with $\|\mathbf{v}_{(t)}\|_{L^2(\nu_{(t)})} \in L^1(0, \tau)$, such that

$$\begin{cases} \frac{d}{dt} \nu_{(t)} + \nabla \cdot (\tilde{J} \mathbf{v}_{(t)} \nu_{(t)}) = 0, & \nu_{(0)} = \nu_0, & t \in (0, \tau) \\ \mathbf{v}_{(t)} \in T_{\nu_{(t)}} \mathcal{P}_{ac}^2(\mathbb{R}^3) \cap \partial H(\nu_{(t)}) & \text{for a.e. } t. \end{cases} \quad (5.12)$$

We now consider Hamilton flows, as in the definition (5.4) and the condition (H1), (H2), (H3) given in section 3.4. The main result on these flows, which is used in our proof of the main theorem 3.6, can be stated as follows:

Theorem 5.4. *Assume that (H1) and (H2) hold for $H(\nu)$ and that $\tau > 0$ satisfies*

$$C_0 \tau \sqrt{24(1 + e^{(25C_0^2+1)\tau}(1 + M_2(\nu_0)))} < R_0. \quad (5.13)$$

Then there exists a Hamiltonian flow $\nu_{(t)} \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$, $\nu_{(t)} : [0, \tau] \rightarrow D(H)$ starting from $\nu_0 \in \mathcal{P}_{ac}^2(\mathbb{R}^3)$, satisfying (5.12), such that the velocity field $\mathbf{v}_{(t)}$ coincides with $\partial_0 H(\nu_{(t)})$ for a.e. $t \in [0, \tau]$. Furthermore, the function $t \rightarrow \nu_{(t)}$ is Lipschitz continuous. Finally, there exists a function $l(r)$ depending only on τ and C_0 such that

$$\nu_0 \geq m_r \text{ a.e. on } B_r \text{ for all } r > 0 \implies \nu_{(t)} \geq m_{l(r)} \text{ a.e. on } B_r \text{ for all } r > 0 \quad (5.14)$$

and

$$\nu_0 \leq M_r \text{ a.e. on } B_r \text{ for all } r > 0 \implies \nu_{(t)} \leq M_{l(r)} \text{ a.e. on } B_r \text{ for all } r > 0. \quad (5.15)$$

If in addition (H3) holds, then $t \mapsto H(\nu_{(t)})$ is constant.

Remark 5.5. Existence of R_0 in the global time condition (5.13) is guaranteed by the fact that ν is compactly supported. This is a crucial property for all our applications.